Numerical Methods for Differential Equations

Chapter 6: PDEs – Waves and hyperbolics

Tony Stillfjord, Gustaf Söderlind

Numerical Analysis, Lund University

LUND UNIVERSITY

Contents

1. Hyperbolic problems

Wave equations with applications

$u_{tt} = u_{xx}$	wave equation *acoustics*
$u_t + cu_x = 0$	linear conservation law *fluids; traffic density*
$u_t + uu_x = 0$	nonlinear conservation law *waste water management*
$u_t + uu_x = u_{xx}$	viscous Burgers equation *seismics* (parabolic wave equation)

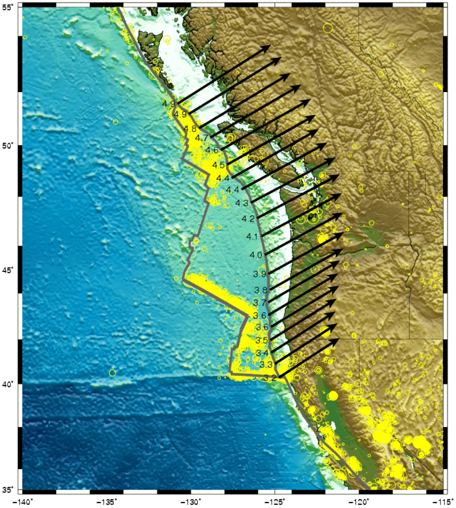
Wave equations with applications...

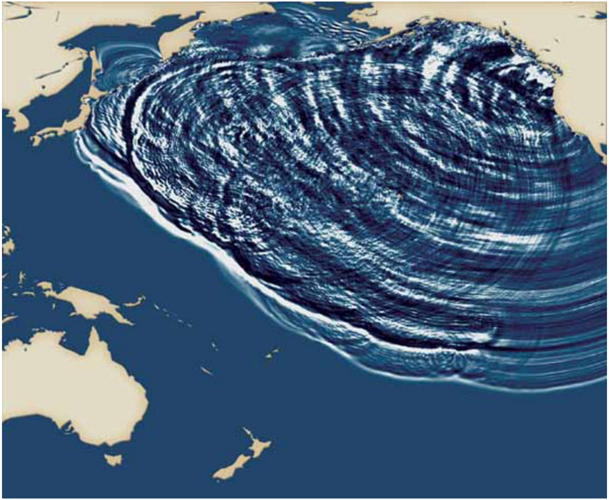
$u_t + uu_x = -u_{xxx}$ Korteweg–de Vries' (KdV) equation
soliton waves

$u_{tt} = u_{xx} - u$ Klein–Gordon equation
telegraph equation
quantum theory

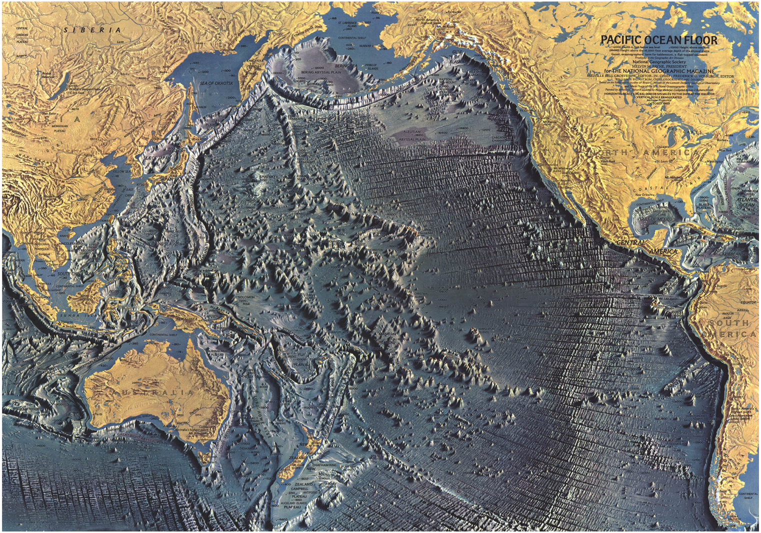
$\mathrm{i}u_t = u_{xx}$ Schrödinger equation
quantum theory

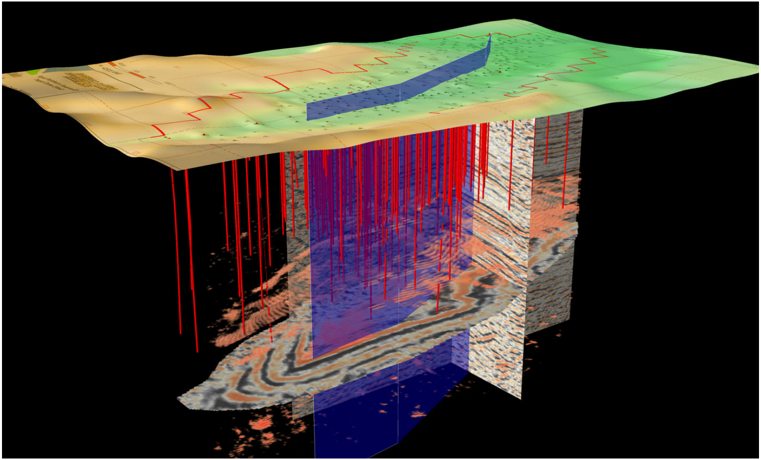
$u_{tt} = -u_{xxxx}$ beam equation
elastic vibrations

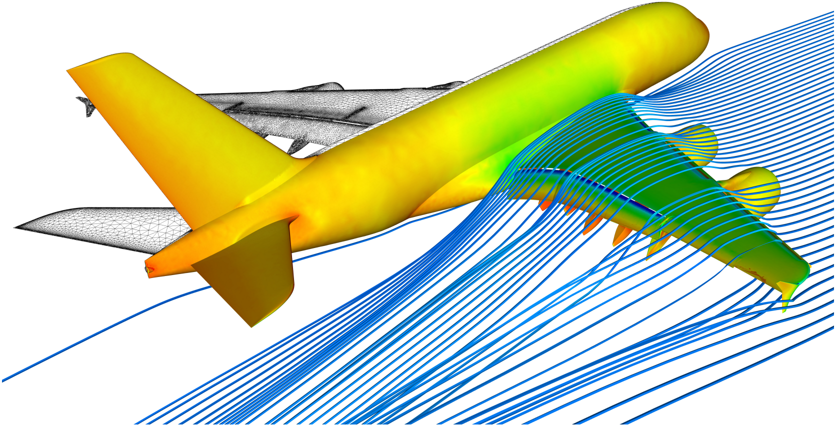

55°
50°
45°
40°
35°
−140°
−135°
−130°
−125°
−120°
−115°
4.9
4.9
4.8
4.7
4.6
4.5
4.4
4.4
4.3
4.2
4.1
4.0
3.9
3.8
3.7
3.6
3.6
3.5
3.4
3.3
3.2



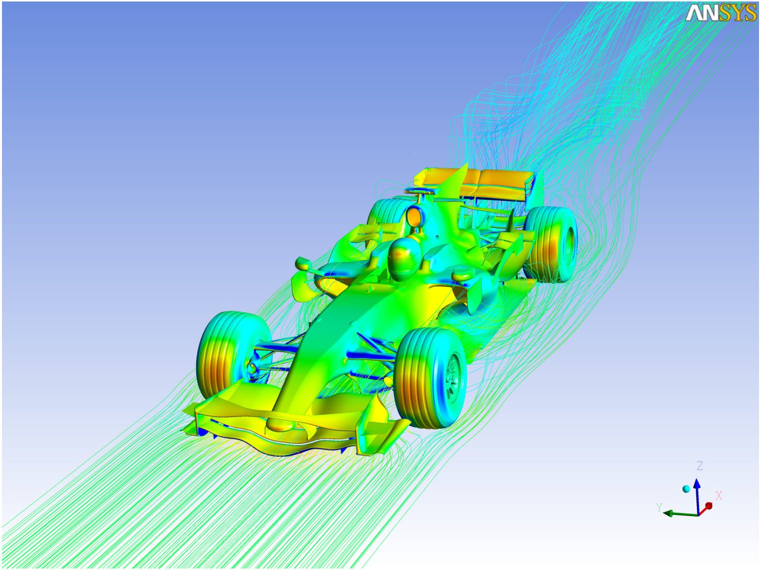
Pacific Ocean floor topography







Computational fluid dynamics *RB4 F1 car (2008)*



Standard hyperbolic model problems

Wave equation $u_{tt} = c^2 u_{xx}$ or *Advection equation* $u_t + cu_x = 0$

Conservation law $u_t + (f(u))_x = 0$ (inviscid flow)

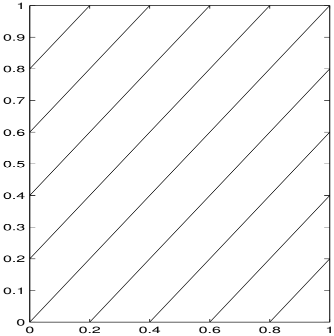
d'Alembert solution $u(t,x) = g(x - ct)$ solves $u_t + cu_x = 0$,
because $u_t = -c \cdot g'$ and $u_x = g'$

Solution u is constant on the characteristics $x - ct = \mathrm{const.}$

The characteristics are straight lines in the solution domain

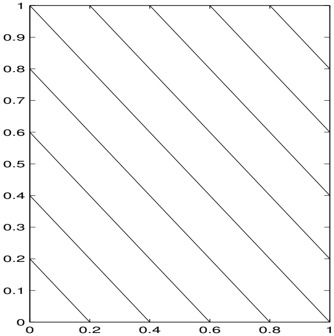
Boundary conditions for $u_t + cu_x = 0$

$c > 0$



Bdry cond's at $x = 0$

$c < 0$



Bdry cond's at $x = 1$

Initial conditions are always required

Let $u(t,x)$ be car density (per unit distance)

Then total number of cars on road segment $[a,b]$ is

$$N(t) = \int_a^b u \,\mathrm{d}x$$

Non-overtaking cars at speed v gives car flux $vu(t,x)$

Hence $\dot{N} = \int u_t \,\mathrm{d}x = \text{influx} - \text{outflux}$, implying

$$\int_a^b u_t \,\mathrm{d}x = vu(t,a) - vu(t,b) = -\int_a^b \frac{\mathrm{d}(vu)}{\mathrm{d}x} \,\mathrm{d}x$$

Traffic flow conservation law

If v is constant, $(vu)' = vu_x$. Hence *conservation law*

$$\int_a^b u_t + vu_x \, \mathrm{d}x = 0 \quad \Rightarrow \quad u_t + vu_x = 0$$

Application "Green wave" traffic light control

More advanced model — if speed depends on u, then $(v(u)u)' = v'(u)u_x + v(u)u_x$. *Nonlinear conservation law*

$$u_t + \big(v(u) + v'(u)\big)u_x = 0$$

Interesting phenomena: *Stau*, pile-ups (shock waves), &c.

2. The advection equation

$$u_t + vu_x = 0$$

Relation to the wave equation $u_{tt} = c^2 u_{xx}$

Factorize the differential operator

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \cdot \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right)$$

General d'Alembert solution for wave equation

$$u(t, x) = G_1(x + ct) + G_2(x - ct)$$

Waves going both left and right

Method of lines semi-discretization (SD) of $u_t + u_x = 0$

$$\dot{u}_l(t) + \frac{1}{\Delta x}\sum_{k=-\alpha}^{\beta} a_k u_{l+k}(t) = 0$$

1. $u_x \approx (u_l^n - u_{l-1}^n)/\Delta x$ *backward difference*
2. $u_x \approx (u_{l+1}^n - u_l^n)/\Delta x$ *forward difference*
3. $u_x \approx (u_{l+1}^n - u_{l-1}^n)/(2\Delta x)$ *symmetric difference*
4. $u_x \approx (\frac{1}{4}u_{l+1}^n + \frac{5}{6}u_l^n - \frac{3}{2}u_{l-1}^n + \frac{1}{2}u_{l-2}^n - \frac{1}{12}u_{l-3}^n)/\Delta x$

Upwind/downwind schemes

Upwind schemes use more points on the side that the information is flowing from

For $u_t + cu_x = 0$ with $c > 0$ information flows left$\rightarrow$right:

1. $u_x \approx (u_l^n - u_{l-1}^n)/\Delta x$ upwind
2. $u_x \approx (u_{l+1}^n - u_l^n)/\Delta x$ downwind
3. $u_x \approx (\frac{1}{2}u_{l+1}^n - \frac{1}{2}u_{l-1}^n)/\Delta x$ symmetric
4. $u_x \approx (\frac{1}{4}u_{l+1}^n + \frac{5}{6}u_l^n - \frac{3}{2}u_{l-1}^n + \frac{1}{2}u_{l-2}^n - \frac{1}{12}u_{l-3}^n)/\Delta x$ upwind

Upwind is necessary for stability

With $c < 0$, scheme 1., 2. and 4. change character

The SD method is of consistent of order p if

$$\frac{1}{\Delta x}\sum_{k=-\alpha}^{\beta} a_k u(t, x + k\Delta x) = u_x(t, x) + \mathrm{O}(\Delta x^p)$$

Using Taylor expansion in *forward shift operator E_x* (see Chap. 2)

Theorem *The SD method is of order p if and only if*

$$a(z) := \sum_{k=-\alpha}^{\beta} a_k z^k = \log z + \mathrm{O}(|z-1|^{p+1}), \quad z \to 1$$

Semidiscretizations for the advection equation

A semidiscretization of order p_1 combined with order p_2 time stepping produces a method of consistency order $p = \min\{p_1, p_2\}$

The structure of the Courant number is $\Delta t/\Delta x \leq C$

$$u_t + u_x = 0 \qquad \rightarrow \qquad \frac{\sum_j c_j u_l^{n+1-j}}{\Delta t} + \frac{\sum_k a_k u_{l+k}^n}{\Delta x} = 0$$

Why is it $\Delta t/\Delta x^2 < C$ *in diffusion equation?*

Construction of FD schemes from SD

Let $\mu = \Delta t/\Delta x$ and consider $u_t + u_x = 0$

1. $u_x \approx (u_l^n - u_{l-1}^n)/\Delta x$ and $u_t \approx (u_l^{n+1} - u_l^n)/\Delta t$ gives the *upwind (Euler) scheme* $u_l^{n+1} = (1-\mu)\, u_l^n + \mu\, u_{l-1}^n$

2. $u_x \approx (u_{l+1}^n - u_l^n)/\Delta x$ and $u_t \approx (u_l^{n+1} - u_l^n)/\Delta t$ gives the *downwind scheme* $u_l^{n+1} = (1+\mu)\, u_l^n - \mu\, u_{l+1}^n$

3. $u_x \approx (u_{l+1}^{n+1} - u_{l-1}^{n+1})/(2\Delta x)$ combined with the explicit midpoint rule $u_t \approx (u_l^{n+2} - u_l^n)/(2\Delta t)$ gives the *leapfrog method* $u_l^{n+2} = \mu\, (u_{l-1}^{n+1} - u_{l+1}^{n+1}) + u_l^n$

3. Classical FD schemes for $u_t + au_x = 0$

1. The *Central difference scheme* (always unstable!)

$$\frac{u_l^{n+1} - u_l^n}{\Delta t} + a\,\frac{u_{l+1}^n - u_{l-1}^n}{2\Delta x} = 0$$

leads to

$$u_l^{n+1} = u_l^n + \frac{a\mu}{2}(u_{l-1}^n - u_{l+1}^n)$$

2. The *Lax–Friedrichs scheme* (convergent, $p = 1$)

$$u_l^{n+1} = \frac{u_{l-1}^n + u_{l+1}^n}{2} + \frac{a\mu}{2}(u_{l-1}^n - u_{l+1}^n)$$

or

$$u_l^{n+1} = \frac{1}{2}(1 + a\mu)\,u_{l-1}^n + \frac{1}{2}(1 - a\mu)\,u_{l+1}^n$$

3. The *Lax–Wendroff scheme* (convergent, $p = 2$)

$$u_l^{n+1} = \frac{a\mu}{2}(1 + a\mu)u_{l-1}^n + (1 - a^2\mu^2)u_l^n - \frac{a\mu}{2}(1 - a\mu)u_{l+1}^n$$

Uses *auto upwinding* dependent on Courant number μ and flow direction a (method coefficients are not symmetric)

4. The *Beam–Warming scheme* (convergent, $p = 2$)

$$u_l^{n+1} = \frac{a\mu}{2}(1 - a\mu)(2 - a\mu)u_l^n + a\mu(2 - a\mu)u_{l-1}^n - \frac{a\mu}{2}(1 - a\mu)u_{l-2}^n$$

Genuine upwind scheme (uses no downwind information)

Derivation of the Lax–Wendroff scheme

$$u(t+\Delta t, x) = u + \Delta t\, u_t + \frac{\Delta t^2}{2} u_{tt} + \dots$$

But $u_t = -a u_x$ implies $\partial_t = -a\partial_x$, hence $u_{tt} = a^2 u_{xx}$ and

$$u(t+\Delta t, x) = u - a\Delta t\, u_x + \frac{a^2 \Delta t^2}{2} u_{xx} + \dots$$

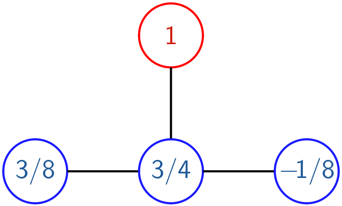
So, with Courant number $\mu = \Delta t / \Delta x$,

$$u_l^{n+1} = \frac{a\mu}{2}(1 + a\mu) u_{l-1}^n + (1 - a^2\mu^2) u_l^n - \frac{a\mu}{2}(1 - a\mu) u_{l+1}^n$$

2nd order accurate as it picks up the first three Taylor terms

Lax–Wendroff computational stencil at $a\mu = 1/2$

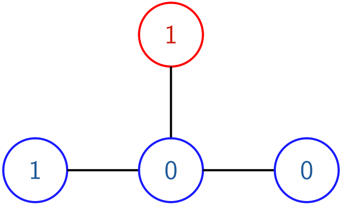
Participating mesh points



Note Asymmetric coefficients correspond to *upwinding*

Lax–Wendroff computational stencil at $a\mu = 1$

Participating mesh points

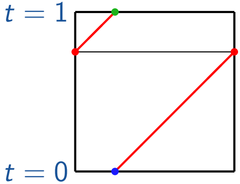


Note *Information transportation along characteristic*

At $a\mu = 1$ the Lax–Wendroff scheme solves $u_t + au_x$ exactly

4. Periodic boundary conditions $u_t + u_x = 0$

In wave phenomena we often have periodicity in t and x. Periodic boundary conditions are defined by $u(t,0) = u(t,1)$ for all $t \geq 0$



$$u(0,x) = g(x) \Rightarrow u(1,x) = g(x)$$

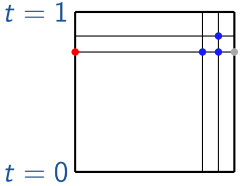
Dynamics on a torus. Stability analysis is relatively simple

Fourier – von Neumann stability analysis

Lax–Wendroff Toeplitz matrix with periodic conditions

$u^{n+1} = A(a\mu)u^n$ where $A(a\mu)$ is a *circulant matrix*

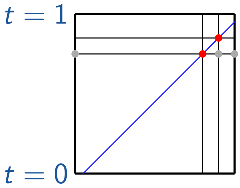
$$A(a\mu) = \begin{pmatrix} 1-a^2\mu^2 & \frac{a\mu}{2}(a\mu-1) & & \frac{a\mu}{2}(a\mu+1) \\ \frac{a\mu}{2}(a\mu+1) & 1-a^2\mu^2 & \frac{a\mu}{2}(a\mu-1) & \\ & \ddots & & \\ \frac{a\mu}{2}(a\mu-1) & & \frac{a\mu}{2}(a\mu+1) & 1-a^2\mu^2 \end{pmatrix}$$



$$u(0,x) = g(x) \Rightarrow u(1,x) = g(x)$$

Taking $a\mu = 1$, the matrix $A(1)$ becomes a *cyclic permutation*

$$A(1) = \begin{pmatrix} 0 & 0 & & 1 \\ 1 & 0 & 0 & \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad u_l^{n+1} = u_{l-1}^n$$



Exact solution along characteristic

$$u(t + \Delta t, x) = u(t, x - \Delta x) = g(x)$$

Lax–Wendroff with periodic conditions $a\mu = 1$

- All permutation matrices P are *orthogonal*, i.e., $P^{-1} = P^{\mathrm{T}}$
- Therefore $A(1)^{\mathrm{T}}A(1) = I$, so $\|A(1)\|_2 = \|A(1)^{-1}\|_2 = 1$
- Hence the Lax–Wendroff method is *stable* at $a\mu = 1$
- The eigenvalues of $A(1)$ are $|\lambda_k[A(1)]| \leq 1$
- By the same token, $|\lambda_k[A(1)^{-1}]| = 1/|\lambda_k[A(1)]| \leq 1$
- Simple unimodular eigenvalues, $\lambda_k[A(1)] = \mathrm{e}^{2\pi\mathrm{i}k/N}$, $k = 1 : N$
- In forward or reverse time $\|u_\cdot^n\|_{\Delta x} = \|u_\cdot^0\|_{\Delta x}$ for all n

This is in agreement with the conservation properties of $u_t = au_x$

$$\int_0^1 uu_t \,\mathrm{d}x = a \int_0^1 uu_x \,\mathrm{d}x \quad \Rightarrow \quad \frac{1}{2}\frac{\mathrm{d}\|u(t,\cdot)\|_{L^2}^2}{\mathrm{d}t} = a \cdot \langle u, u_x \rangle$$

Integrate by parts, using periodic boundary conditions

$$\langle u, u_x \rangle = -\langle u_x, u \rangle = -\langle u, u_x \rangle = 0$$

So u_x is always orthogonal to u, and for all t

$$\frac{\mathrm{d}\|u(t,\cdot)\|_{L^2}^2}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \|u(t,\cdot)\|_{L^2} = \mathrm{const.}$$

Circulant matrices

Structure of circulant matrices

$$C(\kappa) = \begin{pmatrix} \kappa_0 & \kappa_1 & \cdots & \kappa_{d-1} \\ \kappa_{d-1} & \kappa_0 & \cdots & \kappa_{d-2} \\ \vdots & & & \vdots \\ \kappa_1 & \kappa_2 & \cdots & \kappa_0 \end{pmatrix}$$

Circulant matrices are a special class of Toeplitz matrices

(Same element along each diagonal)

Eigenvalues of circulant matrices

Theorem *The eigenvalues of an $N \times N$ circulant matrix C are*

$$\lambda_k[C] = \sum_{j=0}^{N-1} \kappa_j \mathrm{e}^{2kj\pi\mathrm{i}/N}$$

A finite difference scheme with periodic boundary conditions is stable if and only if

$$|\lambda_k[A(a\mu)]| \leq 1$$

Fourier – von Neumann stability

5. The wave equation

$$u_{tt} = u_{xx}, \qquad 0 \leq x \leq 1, \quad t \geq 0$$

Initial conditions $u(0,x) = g_0(x), \quad u_t(0,x) = g_1(x)$
Dirichlet conditions $u(t,0) = \phi_0(t), \quad u(t,1) = \phi_1(t)$

Can be rewritten as a 1st order system $v_t + Av_x = 0$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \lambda[A] = \pm 1$$

Direct semidiscretization

$$\ddot{u}_l - \frac{1}{\Delta x^2} \sum_{k=-\alpha}^{\beta} a_k u_{l+k} = 0$$

The wave equation

Theorem *The direct SD method is of order p if and only if*

$$\sum_{k=-\alpha}^{\beta} a_k z^k = (\log z)^2 + \mathrm{O}(|z-1|^{p+2}),\ z \to 1$$

Analogous to first order case but the method for second order (in time) equations must approximate $(\log z)^2$

Methods for the wave equation

For a linear system of equations a one-sided upwind method cannot be used if eigenvalues have different signs

What is “upwind” is determined by eigenvalue signs (which determine the slopes of the characteristics)

Example $u_t + Au_x = 0, \quad A = V\Lambda V^{-1}, \quad \Lambda = \mathrm{diag}(1, -1)$

May use Lax–Friedrichs and Lax–Wendroff, but not upwind Euler

The vector advection equation

For $u_t + a u_x = 0$, stability must be guaranteed by $a\mu$

For a linear system of advection equations

$$u_t + A u_x = 0, \qquad 0 \leq x \leq 1, \quad t \geq 0$$

with matrix $A = V \Lambda V^{-1}$ with *real eigenvalues* $\lambda_k[A]$,

$$\begin{aligned} u_t + V\Lambda V^{-1} u_x &= 0 \\ V^{-1} u_t + \Lambda V^{-1} u_x &= 0 \\ w_t + \Lambda w_x &= 0 \end{aligned}$$

$\mu\lambda_k[A]$ *must satisfy the CFL stability condition*

Direct semi- and full discretization of the wave equation

2nd order central difference $u_{xx} \approx (u_{l-1} - 2u_l + u_{l+1})/\Delta x^2$ gives symmetric semidiscretization

$$\ddot{u}_l = (u_{l-1} - 2u_l + u_{l+1})/\Delta x^2$$

equivalent to the ODE system

$$\begin{aligned} \dot{u} &= v & u_0 &= u(0,x) \\ \dot{v} &= f(t,u) & w_0 &= \dot{u}(0,x) \end{aligned}$$

Use explicit Euler/implicit Euler combination

$$\begin{aligned} u^{n+1} &= u^n + \Delta t \, v^n \\ v^{n+1} &= v^n + \Delta t \, f(t_{n+1}, u^{n+1}) \end{aligned}$$

The Störmer method

Eliminate v to get a *Störmer method* ($p = 2$)

$$u^{n+2} - 2u^{n+1} + u^n = \Delta t^2\, f(t_{n+1}, u^{n+1})$$

and apply to the SD $\ddot{u}_l = \dfrac{u_{l-1} - 2u_l + u_{l+1}}{\Delta x^2}$

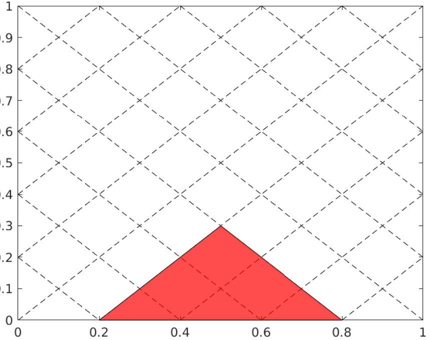
With $\mu = \Delta t / \Delta x$ we get the 2nd order *leapfrog scheme*

$$u_l^{n+2} - 2u_l^{n+1} + u_l^n = \mu^2 (u_{l-1}^{n+1} - 2u_l^{n+1} + u_{l+1}^{n+1})$$

which is stable with Dirichlet conditions for $0 < \mu \leq 1$

Note the 2nd-order time approximation $\frac{u^{n+2} - 2u^{n+1} + u^n}{\Delta t^2} \approx \ddot{u}(t_{n+1})$

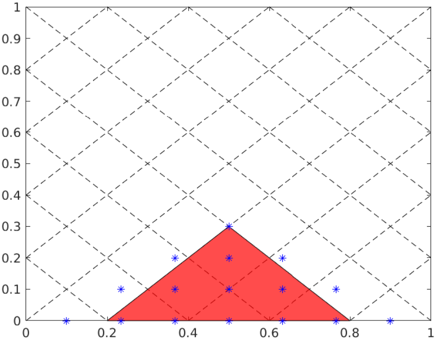
Domain of dependence



Explicit time stepping gives *CFL condition* $\frac{\Delta t}{\Delta x} \lessapprox 1$

Numerical domain of dependence must cover the physical (and hence mathematical) domain of dependence

Domain of dependence

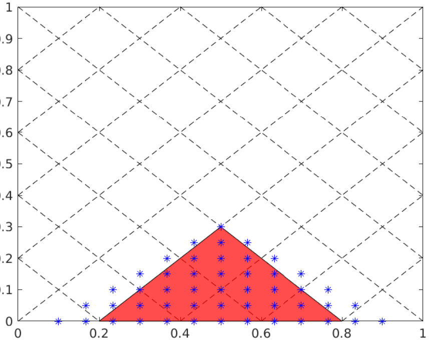


$$\frac{\Delta t}{\Delta x} = 0.75$$

Explicit time stepping gives *CFL condition* $\dfrac{\Delta t}{\Delta x} \lessapprox 1$

Numerical domain of dependence must cover the physical (and hence mathematical) domain of dependence

Domain of dependence

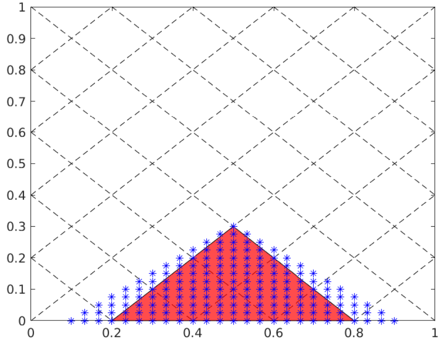


$$\frac{\Delta t}{\Delta x} = 0.75$$

Explicit time stepping gives *CFL condition* $\frac{\Delta t}{\Delta x} \lessapprox 1$

Numerical domain of dependence must cover the physical (and hence mathematical) domain of dependence

Domain of dependence

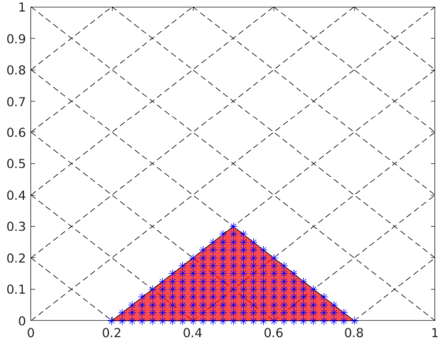


$$\frac{\Delta t}{\Delta x} = 0.75$$

Explicit time stepping gives *CFL condition* $\dfrac{\Delta t}{\Delta x} \lessapprox 1$

Numerical domain of dependence must cover the physical (and hence mathematical) domain of dependence

Domain of dependence

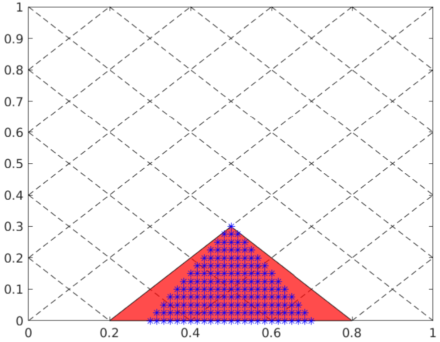


$$\frac{\Delta t}{\Delta x} = 1.0$$

Explicit time stepping gives *CFL condition* $\dfrac{\Delta t}{\Delta x} \lessapprox 1$

Numerical domain of dependence must cover the physical (and hence mathematical) domain of dependence

Domain of dependence



$$\frac{\Delta t}{\Delta x} = 1.5$$

Explicit time stepping gives *CFL condition* $\dfrac{\Delta t}{\Delta x} \lessapprox 1$

Numerical domain of dependence must cover the physical (and hence mathematical) domain of dependence

6. The inviscid Burgers equation

Nonlinear conservation law

$$u_t + (\frac{u^2}{2})_x = u_t + uu_x = 0$$

with $u(0,x) = g(x)$ and $\|g\|_2^2 < \infty$ on $[-\infty, \infty]$

This solution is implicitly characterized by a d'Alembert solution

$$u(t,x) = g(x - ut)$$

for which

$$u_t = -u \cdot g' \,; \quad u_x = g' \quad \Rightarrow \quad u_t + uu_x = 0$$

Inviscid Burgers. . .

Nonlinear conservation law

$$u_t + uu_x = 0\,; \quad u(0,x) = g(x)$$

Take t, x such that $x - ut = \mathrm{const.}$ Then $\dot{x}(t) = u(t,x)$

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, x(t)) = u_t + u_x \dot{x}(t) = u_t + uu_x = 0$$

Along a characteristic, $\mathrm{d}u/\mathrm{d}t = 0 \Rightarrow$ $u(t,x)$ *is constant*

Characteristics are straight lines with slopes equal to the magnitude of u

Inner product $\langle u, u_t \rangle + \langle u^2, u_x \rangle = 0$, using periodic bdry conditions

$$\langle u^2, u_x \rangle = \int_0^1 u^2 u_x \,\mathrm{d}x = -\int_0^1 2uu_x u \,\mathrm{d}x = -\int_0^1 2u^2 u_x \,\mathrm{d}x$$

Therefore,

$$\langle u^2, u_x \rangle = -2\langle u^2, u_x \rangle = 0$$

So u_x is always orthogonal to u^2, and for all t

$$\frac{\mathrm{d}\|u(t,\cdot)\|_{L^2}^2}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \|u(t,\cdot)\|_{L^2} = \mathrm{const.}$$

In fact...

With periodic boundary conditions, u_x is orthogonal to u^p for all p

$$\langle u^p, u_x \rangle = -\langle p u^{p-1} u_x, u \rangle = -p \langle u^p, u_x \rangle$$

So $(1+p)\langle u^p, u_x \rangle = 0$, and if $p \neq -1$,

$$\langle u^p, u_x \rangle = 0$$

What about $p = -1$? Then (without integrating by parts)

$$\langle u^{-1}, u_x \rangle = \int_0^1 \frac{u_x}{u} \, \mathrm{d}x = \int_{u(0)}^{u(1)} \mathrm{d} \log u = 0$$

Inner product $\langle u, u_t \rangle + \langle u, (f(u))_x \rangle = 0$, using periodic boundary conditions and integration by parts

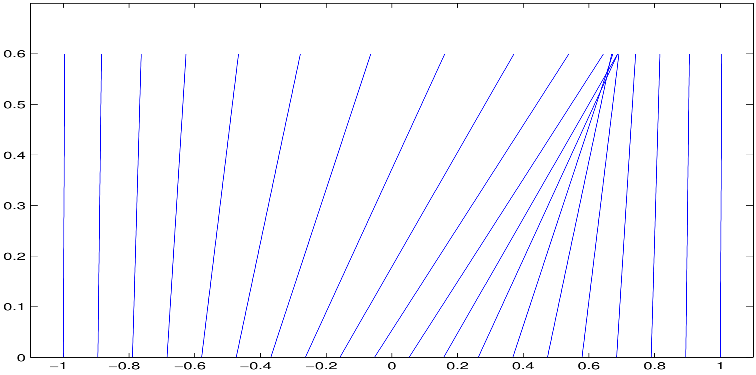
$$\langle u, (f(u))_x \rangle = -\langle u_x, f(u) \rangle = -\int_0^1 \frac{\mathrm{d}u}{\mathrm{d}x} f(u)\,\mathrm{d}x = -\int_{u(0)}^{u(1)} f(u)\,\mathrm{d}u$$

Therefore, $\langle u, (f(u))_x \rangle = 0$ and $\|u(t,\cdot)\|_{L^2} = \mathrm{const.}$ for all $t \geq 0$

The L^2-norm of the solution is conserved

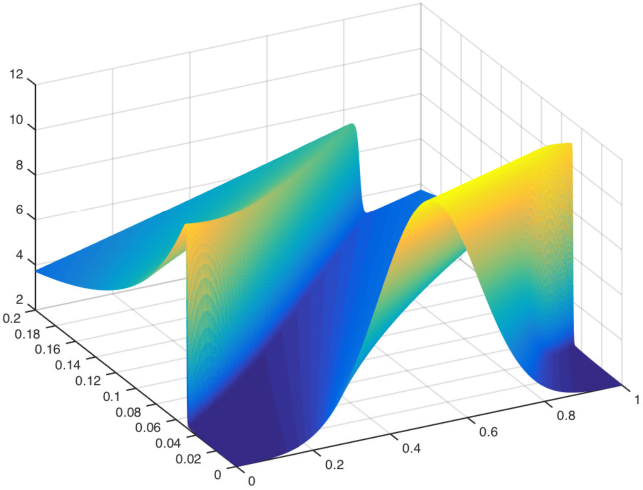
Characteristics $g(x) = \mathrm{e}^{-x^2}$

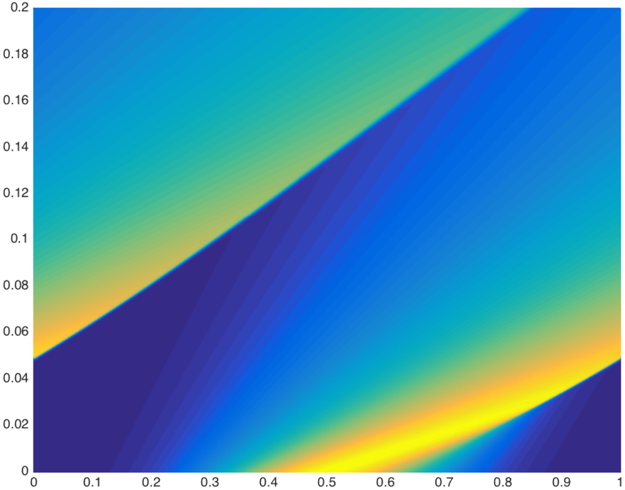
Characteristics may collide, creating a discontinuity (*shock*)



One can determine $u(t, x)$ by following the characteristic

The viscous Burgers equation $u_t + uu_x = \varepsilon\, u_{xx}$





Note Characteristics collide – shock formation (discontinuity)

7. Weak solutions of $u_t + f(u)_x = 0$

Let $\varphi(t,x)$ be any C^1 function with *compact support*

Multiply $u_t + f(u)_x = 0$ by φ and integrate

$$\int_0^\infty \int_{-\infty}^\infty (\varphi u_t + \varphi f_x)\,\mathrm{d}x\,\mathrm{d}t = 0$$

Integrate by parts to see that a solution of the PDE satisfies

$$\int_0^\infty \int_{-\infty}^\infty (\varphi_t u + \varphi_x f(u))\,\mathrm{d}x\,\mathrm{d}t = -\int_{-\infty}^\infty \varphi(0,x)u(0,x)\,\mathrm{d}x$$

Definition $u(t,x)$ is a *weak solution* of $u_t + f(u)_x = 0$ if it satisfies the integral equation above

The Riemann problem

$u_t + f(u)_x = 0 \quad + \quad$ piecewise constant data

Example Burgers inviscid equation $u_t + uu_x = 0$ with

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

The solution depends on relation between u_l and u_r

$$\begin{cases} u_l > u_r & \textit{unique} \text{ weak solution} \\ & \\ u_l < u_r & \textit{infinitely many} \text{ weak solutions, among which} \\ & \text{one is stable with respect to perturbations} \\ & \text{and physically relevant} \end{cases}$$

A *shock* is a discontinuous solution

This occurs for $u_l > u_r$ when

$$u(t,x) = \begin{cases} u_l, & x < s \cdot t \\ u_r, & x > s \cdot t \end{cases}$$

where $s = (u_l + u_r)/2$ is the *shock speed*

Note The shock propagates undamped with speed s in the inviscid Burgers equation

If $u_l < u_r$, there is no shock but a *rarefaction wave* instead

$$u(x,t) = \begin{cases} u_l, & x < u_l t \\ x/t, & u_l t < x < u_r t \\ u_r, & x > u_r t \end{cases}$$

Note The rarefaction wave is continuous but not differentiable.
Characteristics "fan out" instead of colliding

Approximate the solution *locally* by a *piecewise constant function*

$$u_t + f(u)_x = 0\,; \quad u(x,0) = g(x)$$

Define a piecewise constant initial value approximation $w^{[0]}(x,0)$ for $x \in (x_{l-1/2}, x_{l+1/2}]$ with *average*

$$u_l^0 = \frac{1}{\Delta x}\int_{x_{l-1/2}}^{x_{l+1/2}} g(x)\,\mathrm{d}x$$

Thus $w^{[0]}(x,0)$ has a discontinuity at the center of $[x_{l-1}, x_l]$

Godunov method...

With piecewise constant data $w^{[0]}(x, 0)$ in each cell $[x_{l-1}, x_l]$, construct $w^{[0]}(t, x)$ for $0 \le t \le t_1$ by *solving the corresponding Riemann problem exactly*

Define the *approximate solution* at t_1 by taking

$$u_l^1 = \frac{1}{\Delta x} \int_{x_{l-1/2}}^{x_{l+1/2}} w^{[0]}(x, t_1) \, \mathrm{d}x$$

Proceed to construct a *new piecewise constant function* $w^{[1]}(t_1, x) := u_l^1$ for $x \in (x_{l-1/2}, x_{l+1/2}]$

Godunov method...

Cell averages are easy to compute, resulting in

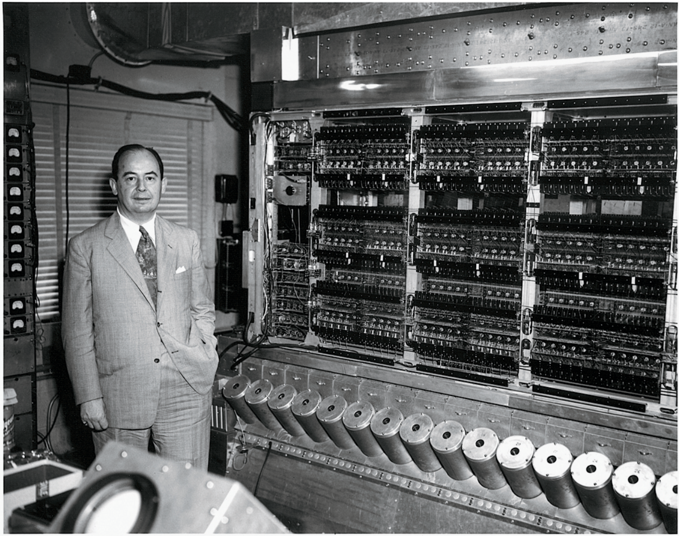
$$u_l^{n+1} = u_l^n - \mu_n[F(u_l^n, u_{l+1}^n) - F(u_{l-1}^n, u_l^n)]$$

where $\mu_n = \Delta t_n / \Delta x$ and the *flux integral*

$$F(u_l^n, u_{l+1}^n) = \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} f(w^{[n]}(x_{l+1/2}, t))\, \mathrm{d}t$$

Note This integral is trivial to compute because $w^{[n]}$ is constant at $x_{l+1/2}$ for $t \in (t_n, t_{n+1})$

John von Neumann 1903 – 1957 *IAS, Princeton 1952*



The end . . . and the beginning!