Numerical Methods for Differential Equations, FMNN10 Pi3, F3
Tony Stillfjord, Gustaf Söderlind

Review questions and study problems, week 6

1. Let Q be a rational function of the form
$$Q(w) = \frac{1 + \alpha w}{\beta + \gamma w}.$$
Then let A be a matrix with known eigenvalues $\lambda_k[A]$. By the matrix $Q(A)$ we mean
$$Q(A) = (\beta I + \gamma A)^{-1}(I + \alpha A).$$
Show that the eigenvalues of this matrix are
$$\lambda_k[Q(A)] = Q(\lambda_k[A]).$$

2. Let the semi-discretization $\dot{u} = T_{\Delta x} u$ be solved using the time discretization
$$(I - \frac{\Delta t}{2} T_{\Delta x}) u_{n+1} = (I + \frac{\Delta t}{2} T_{\Delta x}) u_n.$$
This is the trapezoidal rule, or the Crank–Nicolson method for the diffusion equation $u_t = u_{xx}$. Solving for u_{n+1}, it is equivalent to an explicit recursion
$$u_{n+1} = B(\Delta t, \Delta x) u_n$$
Use the result from the previous question, and your knowledge of $\lambda_k[T_{\Delta x}]$, to find the eigenvalues of $B(\Delta t, \Delta x)$.

   Note that the recursion above can be viewed as a fixed point iteration. State a condition on the eigenvalues for stability (convergence). Given the properties of $\lambda_k[T_{\Delta x}]$, is there a restriction on Δt in order to have a contraction?

3. The semi-discretization $\dot{u} = S_{\Delta x} u$ of the linear hyperbolic conservation law $u_t = u_x$ is given. Consider the three temporal discretizations

   (a) Explicit Euler $u_{n+1} = (I + \Delta t S_{\Delta x}) u_n$
   (b) Implicit Euler $u_{n+1} = (I - \Delta t S_{\Delta x})^{-1} u_n$
   (c) Trapezoidal rule $u_{n+1} = (I - \frac{\Delta t}{2} S_{\Delta x})^{-1}(I + \frac{\Delta t}{2} S_{\Delta x}) u_n$

   Find the eigenvalues that govern stability in each of the three cases, and find what condition on Δt is necessary for stability in each case.

4. Given that the conservation law has *no growth* and *no decay*, and can be solved in forward as well as reverse time, which one of the three discretizations would you prefer for this problem, on the grounds that it replicates a behavior similar to that of the original PDE?

   (**Hint:** What conditions do you need to impose on the eigenvalues in order to have no growth in forward time and no growth in reverse time?)

5. Now consider the "Leap-frog" scheme $u_{n+1} = u_{n-1} + 2\Delta t S_{\Delta x} u_n$ for the conservation law. As this is a two-step (i.e., multistep) method, it is more difficult to analyze its stability. In order to get acquainted with its stability properties, apply the method to the linear test equation $\dot{y} = \lambda y$ and

   (a) Find the characteristic equation of this scalar recursion

   (b) Show that the product of the two roots is -1

   (c) Show that if any one of the characteristic roots is less than 1, then the method is unstable.

   (d) Show that the stability region is the open interval $\Delta t \lambda \in (-\mathrm{i}, \mathrm{i})$.

6. Next study the Leap-frog method applied to the the semi-discretization $\dot{u} = S_{\Delta x} u$. We know the eigenvalues and eigenvectors of $S_{\Delta x}$, and write

$$S_{\Delta x} U = U \Lambda,$$

   where U is the matrix of eigenvectors, and Λ is the diagonal matrix of the corresponding eigenvalues.

   Now put $u_n = U v_n$. Show that after this transformation the recursion reduces to a scalar recursion (one for each eigenvalue),

$$w_{n+1} = w_{n-1} + 2\Delta t \lambda_k [S_{\Delta x}] w_n$$

   (a) Find the CFL condition on Δt such that the method is stable.

   (b) If the method would be run in reverse, while the CFL stability condition is fulfilled, does the method replicate the problem's behavior?

7. Let us return to the linear hyperbolic conservation law $u_t = u_x$ and to the discretization $u_{n+1} = (I + \Delta t S_{\Delta x}) u_n$. This is the *central difference scheme* which you found out to be *unstable* in Computer Project 3. Now, we know that the eigenvalues of the matrix $S_{\Delta x}$ are

$$\lambda_k = \mathrm{i}\omega_k / \Delta x$$

for $k = 1, \dots K$, where $K \to \infty$ as $\Delta x \to 0$. Using the central difference scheme to solve up to time $t = n\Delta t$, using n steps, implies that we are interested in the limit

$$\lim_{n\to\infty} \left(1 + \frac{\mathrm{i}\omega_k \Delta t}{\Delta x}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\mathrm{i}\omega_k t/\Delta x}{n}\right)^n.$$

What is this limit when Δx is finite? Why does it break down as $\Delta x \to 0$?

8. Derive the Lax–Wendroff method for $u_t + au_x = 0$, using Taylor series expansion.

9. The Lax–Friedrichs scheme is

$$u_j^{n+1} = (u_{j+1}^n + u_{j-1}^n)/2 - a\Delta t(u_{j+1}^n - u_{j-1}^n)/(2\Delta x)$$

It is claimed to be stable up to CFL $= |a\Delta t/\Delta x| = 1$. Construct the Toeplitz circulant matrix associated with this method when applied to a problem with periodic boundary conditions. In particular, write down the matrix at CFL $= 1$. Is the matrix symmetric, skew-symmetric or unsymmetric at this point? Give an interpretation of this particular matrix (i.e., what does it do to a vector?). Determine its eigenvalues analytically at CFL $= 1$. Is the method stable there? Motivate why it is of particular interest to run the matrix on the CFL limit.