Numerical Methods for Differential Equations, FMNN10 Pi3, F3
Tony Stillfjord, Gustaf Söderlind

Review questions and study problems, week 5

A problem is said to be *well posed* if it has a unique solution which depends *continuously* on the data. For example, in the simple 2p-BVP $u'' = q(x)$ with $u(0) = u(1) = 0$ that we studied in the BVP part of the course, we found, by using the logarithmic norm, that

$$\|u\|_{\Delta x} \leq \frac{\|q\|_{\Delta x}}{\pi^2}.$$

Hence, if the *data* $\|q\|_{\Delta x} \to 0$, then the *solution* $\|u\|_{\Delta x} \to 0$. This implies continuity: a small change in the data q has only a bounded effect on the solution u. The problem is *well posed.*

In a similar way, an initial value problem $\dot{u} = f(u)$ with $u(0) = u_0$ is well posed if the solution depends continuously on the data u_0. A small change in the initial value u_0 must only have a bounded effect on the solution $u(t)$. Note that here it's enough that the effect is bounded at a finite time t. So a problem like

$$\dot{u} = u; \quad u(0) = 1$$

is well posed: if we perturb $u(0)$ by ε the solution $u(t)$ changes by $\mathrm{e}^t\varepsilon$, which is bounded for any finite t.

There are several questions concerning well-posedness below. They cannot be treated in a complete way here, so we will only take a simplified approach, looking at a continuous data dependence. Thus, you don't have to address existence questions below. Moreover, as boundary conditions also affect well-posedness we will for simplicity disregard the influence of boundary conditions below. This corresponds to either having homogeneous boundary conditions, periodic boundary conditions, or "none at all" when the computational domain is infinite, i.e., $x \in (-\infty, \infty)$.

1. Show that the diffusion equation $u_t = u_{xx}$ with boundary conditions $u(t, 0) = u(t, 1) = 0$ and initial value $u(0, x) = g(x)$ is well posed for $t \geq 0$.

   (**Hint:** Using the log norm of $\partial^2/\partial x^2$, how large is the influence of a perturbation of g? Use the same technique as we used for the initial value problem above.)

2. Show that the diffusion equation $u_t = u_{xx}$ is not well posed in *reverse time*, i.e., for $t < 0$.

   (**Hint:** Use the same technique as above, but note that you will need the log norm of $-\partial^2/\partial x^2$, as the reversed problem is equivalent to solving $u_t = -u_{xx}$ in forward time.)

3. Consider the semi-discretization (method of lines) $\dot{u} = T_{\Delta x} u$ of the diffusion equation. If the initial condition is changed by ε, how large is the perturbation at a given time $t > 0$?

4. Same question for the reversed problem, $\dot{u} = -T_{\Delta x} u$. Can you conclude that the semi-discretization is ill posed as $\Delta x \to 0$?

5. (A hard problem) Let the skew-symmetric matrix

$$R = \begin{pmatrix} 0 & 1 & 0 & \dots & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & & \ddots & \\ & \dots & 0 & -1 & 0 \end{pmatrix}$$

   be given, and note that $R_{\Delta x} = R/(2\Delta x)$ is a second order accurate approximation to $\partial/\partial x$.

   (a) Show that the eigenvalues of $R_{\Delta x}$ are purely imaginary,

$$\lambda_k[R_{\Delta x}] = \frac{\mathrm{i}}{\Delta x} \cos \frac{k\pi}{N+1}, \qquad k = 1 : N.$$

   (b) Determine $\mu_2[R_{\Delta x}]$ and $\mu_2[-R_{\Delta x}]$. (**Hint:** Do not try to compute the logarithmic norms. Instead, use the fact that we know that the Euclidean norm is sharp for normal matrices, and hence $\mu_2[R_{\Delta x}] = \max_k \operatorname{Re} \lambda_k[R_{\Delta x}]$.)

   (c) Is the semi-discretization $\dot{u} = R_{\Delta x} u$ well posed in forward time? Is it well posed in reverse time?

6. In view of what you found above, consider the two-point boundary value problem $y'' + Ky' + y = q(x)$ with homogeneous boundary conditions. Does the value of the constant K matter at all for whether the problem is well-posed or not?

7. Consider instead $y'' + y' + Ky = q(x)$ with homogeneous boundary conditions. Does the value of the constant K matter now? (**Hint:** In these problems, look at how the constant K affects the logarithmic norm of the operator on the left-hand side.)

8. Consider the linear hyperbolic conservation law $u_t + u_x = 0$. Show that it is well posed both in forward and reverse time. How much does a perturbation in the initial data grow until time t in forward and reverse time, respectively?

9. (Difficult) Find out whether the *convection–diffusion equation* $u_t = u_x + u_{xx}/\mathrm{Pe}$ is well-posed in forward time. Here $0 < \mathrm{Pe} < \infty$ is the *Peclet number*, which measures the "balance" between convection and diffusion. You can take homogeneous Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$. Differentiate $\|u(t,\cdot)\|_2^2$ and use the Cauchy-Schwarz inequality as well as Young's inequality. The latter says that $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ for any positive numbers a, b and ϵ.

10. Check if the convection–diffusion equation is well posed in reverse time. Which of the two operators, the convection part $\partial/\partial x$, or the diffusion part $\partial^2/\partial x^2$, is decisive for well-posedness? Note that finding an explicit counterexample to well-posedness is not easy.

11. Consider the Laplacian in 2D,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

on the unit square $[0,1] \times [0,1]$ with $u = 0$ on the boundary. Consider the corresponding eigenvalue problem, $\Delta u = \lambda u$. Write $u(x,y) = X(x)Y(y)$ and insert it into the eigenvalue problem. Show that

$$\lambda_{i,j}[\Delta] = \kappa_i[\mathrm{d}^2/\mathrm{d}x^2] + \kappa_j[\mathrm{d}^2/\mathrm{d}y^2],$$

where $\kappa_i[\mathrm{d}^2/\mathrm{d}x^2]$ is the ith eigenvalue of $\mathrm{d}^2/\mathrm{d}x^2$ on $[0,1]$ with boundary conditions $X(0) = X(1) = 0$. What is the largest eigenvalue (note, not in magnitude!) of Δ on the given domain and with the given boundary conditions?

12. Now consider the discrete eigenvalue problem for the five-point finite difference operator approximating the Laplacian,

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = \lambda u_{i,j},$$

using $\Delta x = \Delta y$. Write $u_{i,j} = X_i Y_j$ and try to "separate" the eigenvalue problem like you did in the continuous case. Conclude what the eigenvalues are, using your knowledge of the eigenvalues of standard Toeplitz matrix we use to approximate $\mathrm{d}^2/\mathrm{d}x^2$.

13. In the lecture notes we saw that, using the five-point operator, we can represent the discrete Laplacian (a linear operator) by the matrix

$$L_{\Delta x} = \frac{1}{\Delta x^2} \begin{pmatrix} T & I & 0 & \dots & & \\ I & T & I & & & \\ & I & T & I & & \\ & & & & & \\ & & & & \ddots & I \\ & \dots & & 0 & I & T \end{pmatrix},$$

with $T = \text{tridiag}(1 \; {-4} \quad 1)$. Note that the matrix $L_{\Delta x}$ is *symmetric*, but it is not a Toeplitz matrix. Using the information on the eigenvalues of $L_{\Delta x}$ found in the previous problem, find the Euclidean logarithmic norm $\mu_2[L_{\Delta x}]$.