Numerical Methods for Differential Equations, FMNN10 Pi3, F3
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Review questions and study problems, week 4

1. True or false (justify your answer): *Consider the Sturm-Liouville problem*

$$\frac{d}{dx}\left((1-0.8\sin^2 x)\frac{dy}{dx}\right) - \lambda y = 0, \quad y(0) = y(\pi) = 0.$$

The following discretization

$$\frac{p_{n-1}y_{n-1} - 2p_n y_n + p_{n+1}y_{n+1}}{\Delta x^2} = \lambda_{\Delta x} y_n, \quad n = 1 : N$$

$$y_0 = y_{N+1} = 0$$

where $p_n = 1 - 0.8\sin^2\dfrac{n\pi}{N+1}$ *is of order 2.*

2. Give a 4×4 example of
   - A tridiagonal symmetric Toeplitz matrix
   - A skew-symmetric Toeplitz matrix
   - A lower triangular Toeplitz matrix

3. Solve the linear difference equation

$$\begin{aligned} 6u_{j+2} - 5u_{j+1} + u_j &= 0 \qquad (j = 0 : N-1) \\ u_0 = 1,\ u_{N+1} &= 0. \end{aligned}$$

4. True or false (justify your answer): *If $\lambda[T]$ are the eigenvalues of T and $\lambda[S]$ the eigenvalues of S, then $\lambda[T] + \lambda[S]$ are the eigenvalues of $T + S$.*

5. Let $Au = \lambda u$. Show that A^{-1} has the eigenvalues $1/\lambda$.

6. True or false: *If $Au = \lambda u$, then e^{tA} has the eigenvalues $\mathrm{e}^{t\lambda}$.*

7. In class we determined the eigenvalues of

$$T_{\Delta x} = \frac{1}{\Delta x^2}\mathrm{tridiag}(1 \quad -2 \quad 1)$$

   (a) Sketch the location of the eigenvalues in the complex plane

(b) Sketch the location of the eigenvalues of $T_{\Delta x}^{-1}$. (Make sure that your sketches have some "reasonable scaling," e.g. by indicating where the eigenvalues are in relation to the unit circle.)

(c) If $\Delta x \to 0$, where will the eigenvalues of $T_{\Delta x}^{-1}$ "cluster?"

8. Consider the initial value problem $\dot{u} = T_{\Delta x} u$ with initial condition $u(0) = v$.

   (a) Give an upper bound for $\|e^{tT_{\Delta x}}\|_2$ for $t \geq 0$.

   (b) Sketch the location of the eigenvalues of $e^{tT_{\Delta x}}$ in the complex plane. (You may consider time t to be a fixed parameter.)

   (c) Where do the eigenvalues of $e^{tT_{\Delta x}}$ "cluster" as $\Delta x \to 0$ for t fixed?

   (d) Where do they go as $t \to \infty$ for Δx fixed?

   (e) Can you give or suggest an upper bound for the *inverse* $\|e^{-tT_{\Delta x}}\|_2$ (where $t > 0$)? How does that inverse behave as $\Delta x \to 0$?

   (f) Suppose we solve this initial value problem using the explicit Euler method. What condition on the time step Δt is a minimum requirement for stability?

   (g) Same question for the implicit Euler method.

   (h) Which method is suitable when $\Delta x \to 0$?

9. Consider the diffusion equation $u_t = u_{xx}$ with homogeneous boundary conditions $u(t, 0) = u(t, 1) = 0$ and write

$$\frac{1}{2}\frac{\mathrm{d}\|u\|_2^2}{\mathrm{d}t} = \langle u, u_t \rangle$$

for the usual inner product $\langle v, u \rangle = \int vu \, \mathrm{d}x$. Use integration by parts and Sobolev's lemma (alternatively the logarithmic norm of $\mathrm{d}^2/\mathrm{d}x^2$) to show that

$$\|u(t, \cdot)\|_2 \leq e^{-t\pi^2} \|u(0, \cdot)\|_2$$

10. In class we determined the eigenvalues of symmetric Toeplitz matrix $T = \mathrm{tridiag}(1 \quad 0 \quad 1)$ analytically.

    (Difficult) Determine, with a similar methodology, the eigenvalues of the skew symmetric Toeplitz matrix $S = \mathrm{tridiag}(-1 \quad 0 \quad 1)$.

11. Let y' be approximated by the second order, symmetric difference quotient $S_{\Delta x} = S/(2\Delta x)$. What are the

    (a) eigenvalues of $S_{\Delta x}$

    (b) Euclidean norm of $S_{\Delta x}$

(c) Euclidean logarithmic norm of $S_{\Delta x}$

(**Hint:** In class we showed that the Euclidean norm is "sharp" for symmetric matrices, and those proofs can easily be modified to see that the Euclidean norm is also sharp for skew-symmetric matrices.)

12. Consider the 2pBVP $u'' + u' + u = f(x)$ with $u(0) = u(1) = 0$.

    (a) Find (an upper bound of) the logarithmic norm of the operator

    $$\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x} + 1.$$

    Does the problem have a unique solution for every right-hand side f? (Use the fact that $\mu[A+B] \leq \mu[A]+\mu[B]$ and combine it with the Uniform Monotonicity Theorem.)

    (b) Introduce a suitable grid and discretize the equation above. Use the same techniques as in the previous problem to show that your *discretization* has a unique solution for every right-hand side f.

    (c) Let $u_{\Delta x}$ denote the solution vector on the grid. Give a bound for $\|u_{\Delta x}\|_{\Delta x}$ in terms of $\|f\|_{\Delta x}$ and the logarithmic norm.

13. Consider the 2pBVP $y'' + \omega^2 y = g(x)$ with homogeneous boundary data $y(0) = y(1) = 0$.

    (a) For what values of the parameter ω can you guarantee that there is a unique solution?

    (b) Let $\omega = \pi$. What happens with the analytical solution? Why?