Numerical Methods for Differential Equations, FMNN10 Pi3, F3

Tony Stillfjord, Gustaf Söderlind

Review questions and study problems, week 3

1. Write the definition of the *logarithmic norm.*

2. True or false: *For the Euclidean vector norm,*

$$\mu_2[A] = \max_{x \neq 0} x^{\mathrm{T}} A x / x^{\mathrm{T}} x.$$

3. Using the Uniform Monotonicity Theorem, give a bound for $\|A^{-1}\|_\infty$, given that $\mu_\infty[A] < 0$.

4. Explain which of the following two bounds is sharper, and why:
   - $\|x(t)\| \leq \mathrm{e}^{t\mu[A]} \, \|x(0)\| \, ; \quad t \geq 0$
   - $\|x(t)\| \leq \mathrm{e}^{t\|A\|} \, \|x(0)\| \, ; \quad t \geq 0$

5. Suppose $A = \lambda$, a complex number. Then the "norm" is simply the absolute value, $\|A\| = |\lambda|$. What is the logarithmic norm of λ?

6. Show that

$$\lim_{h \to 0+} \frac{|1 + hz| - 1}{h} = \mathrm{Re}\, z$$

7. Using the results of the previous two problems, consider the linear test equation $\dot{y} = \lambda y$ with $y(0) = 1$. When is $|y(t)| \leq \mathrm{e}^{t|\lambda|}$ a "sharp" bound? And when is the corresponding bound with the logarithmic norm "sharp?"

8. Let E be the *forward shift* operator, Δ the *forward difference* operator, ∇ the *backward difference* operator, M the *forward averaging* operator, $M = (E+1)/2$, W the *backward averaging* operator, $W = (1+E^{-1})/2$ and D the *differential* operator. Describe each operator in terms of the others by filling in the following table

	E	Δ	∇	M	W	hD
E	$\times$					
Δ		$\times$				
∇			$\times$			
M				$\times$		
W					$\times$	
hD						$\times$

9. Consider the two-point boundary value problem

$$y'' = x^2 + y^2 \qquad y(0) = 0,\ y(1) = 0$$

Approximate y'' by $\dfrac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}$ and write the corresponding discretization for this BVP. Take $N = 4$; write the nonlinear system of equations $F(y) = 0$ for the unknowns y_1, y_2, y_3, y_4.

10. What is the Jacobian for the problem above?

11. Once you have the Jacobian, how do you perform one Newton iteration to solve $F(y) = 0$?

12. Consider the two-point boundary value problem

$$y'' = x^2 + y^2 \qquad y(0) = 0,\ y'(1) = 0$$

Approximate y'' by $\dfrac{y_{n-1} - 2y_n + y_{n+1}}{\Delta x^2}$ and write the corresponding discretization for this BVP. Take $N = 4$; write the nonlinear system of equations $F(y) = 0$ for the unknowns y_1, y_2, y_3, y_4. Discretize the Neumann boundary condition so that the resulting method is of second order.