Numerical Methods for Differential Equations, FMNN10 Pi3, F3
Tony Stillfjord, Gustaf Söderlind

Review questions and study problems, week 2

1. Give the ρ polynomial for the k-step Adams method.
2. Give the σ polynomial for the k-step BDF method.
3. True or false (justify your answer): *If ρ and σ are the characteristic polynomials of a consistent multistep method, then $\rho(1) = 0$.*
4. Determine which of the following methods are 0-stable (zero-stable).
   - $y_{n+2} = y_{n+1} + h\left(\frac{5}{12}f_{n+2} + \frac{8}{12}f_{n+1} - \frac{1}{12}f_n\right)$
   - $y_{n+2} = y_n + 2hf_{n+2}$
   - $y_{n+2} = \frac{4}{3}y_{n+1} - \frac{1}{3}y_n + \frac{2}{3}hf_{n+2}$
   - $y_{n+2} = 3y_{n+1} - 2y_n + hf_{n+2}$
5. The following is a method of order 4. To what family does it belong?
$$y_{n+3} = y_{n+2} + h\left(\frac{9}{24}f_{n+3} + \frac{19}{24}f_{n+2} - \frac{5}{24}f_{n+1} + \frac{1}{24}f_n\right)$$
   Is it explicit or implicit?
6. Find the order of the method
$$y_{n+2} = y_n + h\left(\frac{1}{3}f_{n+2} + \frac{4}{3}f_{n+1} + \frac{1}{3}f_n\right)$$
7. Write the formula for BDF3. What is the order of this method? Is it zero-stable? Is it convergent? Justify.
8. True or false:
   - *Backward differentiation formulas of orders between 1 and 6 are 0-stable*
   - *Backward differentiation formulas of orders between 1 and 6 are A-stable*
   - *Backward differentiation formulas of orders between 1 and 6 are a good choice for solving stiff problems*
9. Write down in full the *linear test equation* (together with its exact solution) used to study the stability of methods for a scalar IVP.

10. Define A-stability.
11. What is the difference between Runge–Kutta and multistep methods?
12. True or false (justify your answer): *All explicit Runge-Kutta methods of order 3 are convergent.*
13. Let $\dot{y} = Ay$ have a matrix A that is diagonalized by a transformation matrix T, i.e., $T^{-1}AT = \Lambda$ with $\Lambda = \text{diag}(\lambda_i)$. Write down any method of your choice (e.g. the explicit Euler, or the general form of a multistep method) and

    - apply it to $\dot{y} = Ay$
    - then diagonalize the resulting difference equation

    Do you get the same result as if you had applied your method to the diagonalized system $\dot{z} = \Lambda z$ directly? (Here we assume that $y = Tz$.) In more general words, does discretization and diagonalization commute?
14. Construct the Butcher tableau for the 3-stage Heun method,

$$\begin{aligned} Y_1' &= f(t_n, y_n) \\ Y_2' &= f(t_n + h/3, y_n + hY_1'/3) \\ Y_3' &= f(t_n + 2h/3, y_n + 2hY_2'/3) \\ y_{n+1} &= y_n + h\,(Y_1' + 3Y_3')/4 \end{aligned}$$

15. Write the equations for the Runge-Kutta method with the Butcher tableau

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

    Is this an explicit or implicit method?
16. Suppose you apply the RK4 method to the linear test equation $y' = \lambda y$. You then get $y_{n+1} = P(h\lambda)y_n$, where the polynomial $P(h\lambda)$ is called the *stability function* of the method. Derive $P(h\lambda)$ for the RK4 method by hand. If you look at the polynomial, you probably recognize it. What does the polynomial approximate? Can you explain this?
17. Find the stability function of the Runge-Kutta method given by

$$\begin{array}{c|cc} 1/3 & 1/3 & 0 \\ 2/3 & 1/3 & 1/3 \\ \hline & 1/2 & 1/2 \end{array}$$

Is the method A-stable?

18. What is an *embedded Runge–Kutta method?*

19. Can you give an example of an A-stable explicit Runge-Kutta method? Can you give an example of an A-stable multistep method of order 3? Motivate the answers.

20. Write the trapezoidal rule as an implicit Runge–Kutta method, both in terms of the Butcher tableau and in equations.