8.44. a) Verifiera att

$$f(x,y) = y^2 + 4x^2 - x^4$$

har lok. min i origo.

stationärpunkt

$$\begin{cases} f'_x = 8x - 4x^3 \\ f'_y = 2y \end{cases}$$

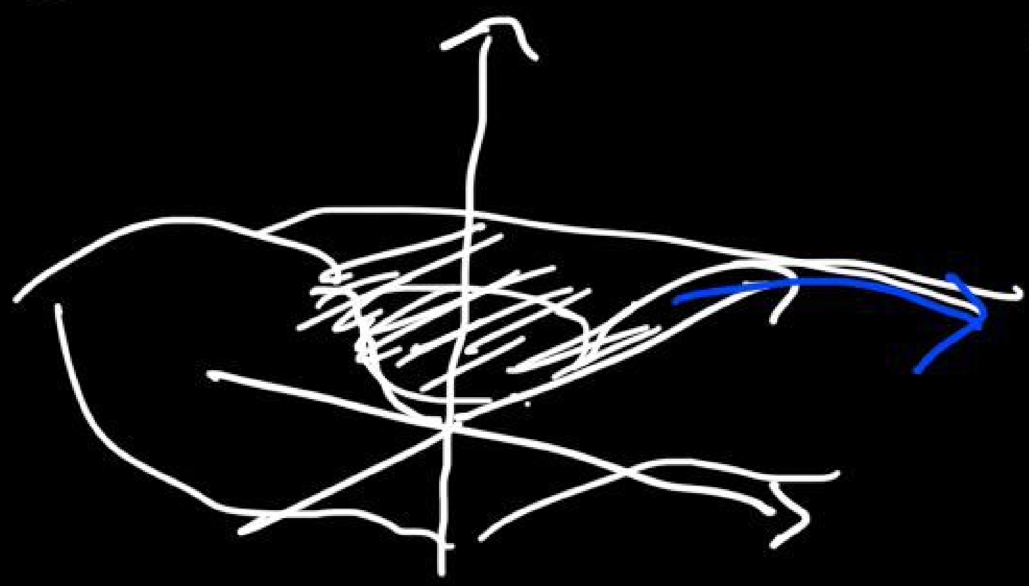
$f'_x(0,0) = f'_y(0,0) = 0$ OK!

$$\begin{cases} f''_{xx} = 8 - 12x^2 \\ f''_{xy} = 0 \\ f''_{yy} = 2 \end{cases}$$

$$Q(h,k) = f''_{xx}h^2 + 2f''_{xy}hk + f''_{yy}k^2 =$$

$$= 8h^2 + 0 + 2k^2 = 8h^2 + 2k^2$$

pos. definit $\Rightarrow$ lok. min.



~~B)~~ b)+c) $\quad f(x,y) = y^2 + 4x^2 - x^4$

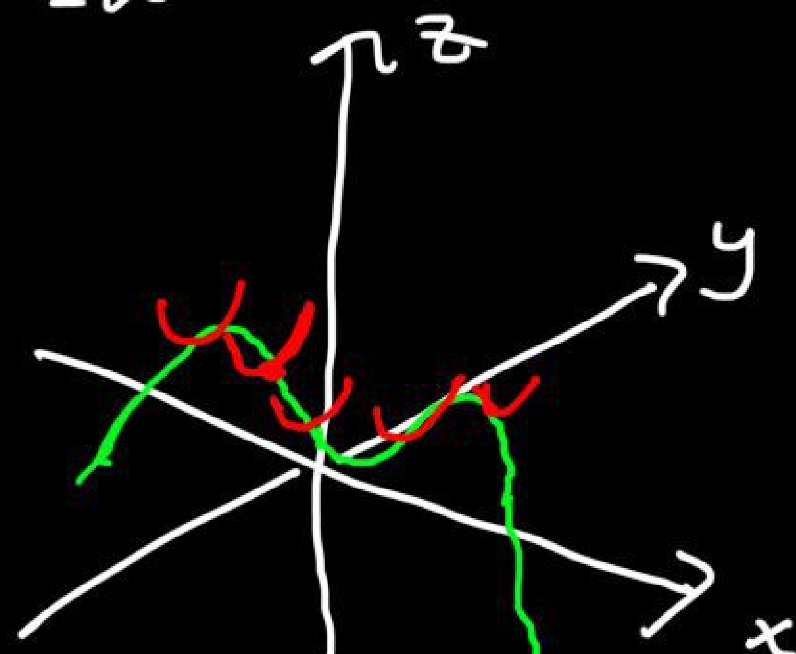
Vi får en "skål" kring origo. Hur stor är volymen av denna?

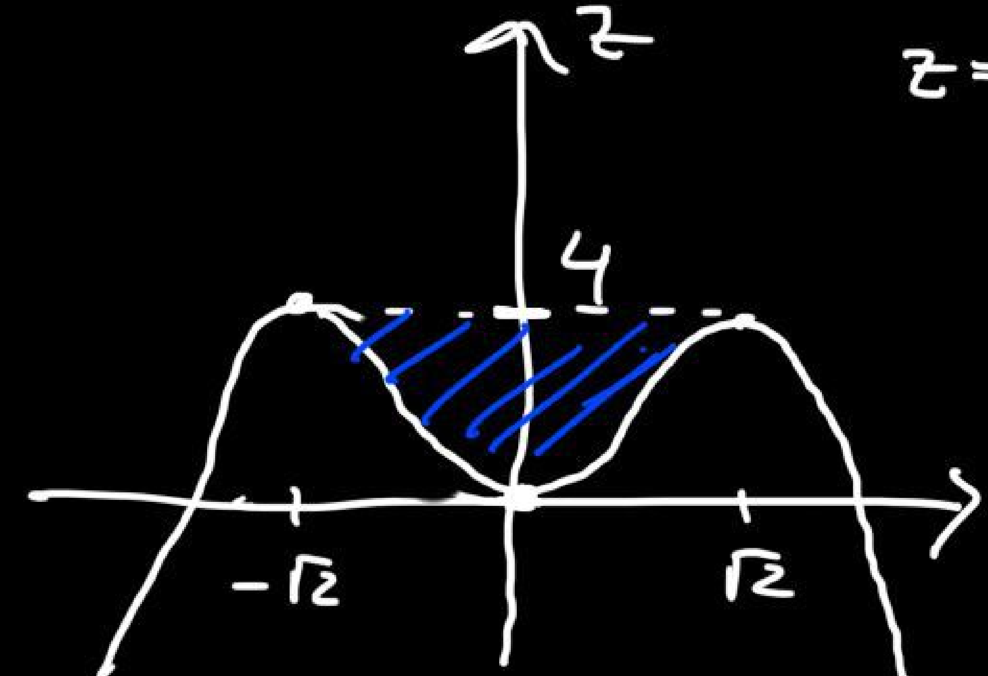
<u>$y=0$</u>: $\quad g(x) = f(x,0) = 4x^2 - x^4$

$$g'(x) = 8x - 4x^3 = -4x(x^2-2) = -4x(x-\sqrt{2})(x+\sqrt{2}) = 0$$

$\iff x = 0$ eller $x = \pm\sqrt{2}$

x		$-\sqrt{2}$		0		$\sqrt{2}$	
g'	$+$	0	$-$	0	$+$	0	$-$
g	$_{-\infty}\nearrow$	4	$\searrow$	0	$\nearrow$	4	$\searrow_{-\infty}$





$z = g(x)$

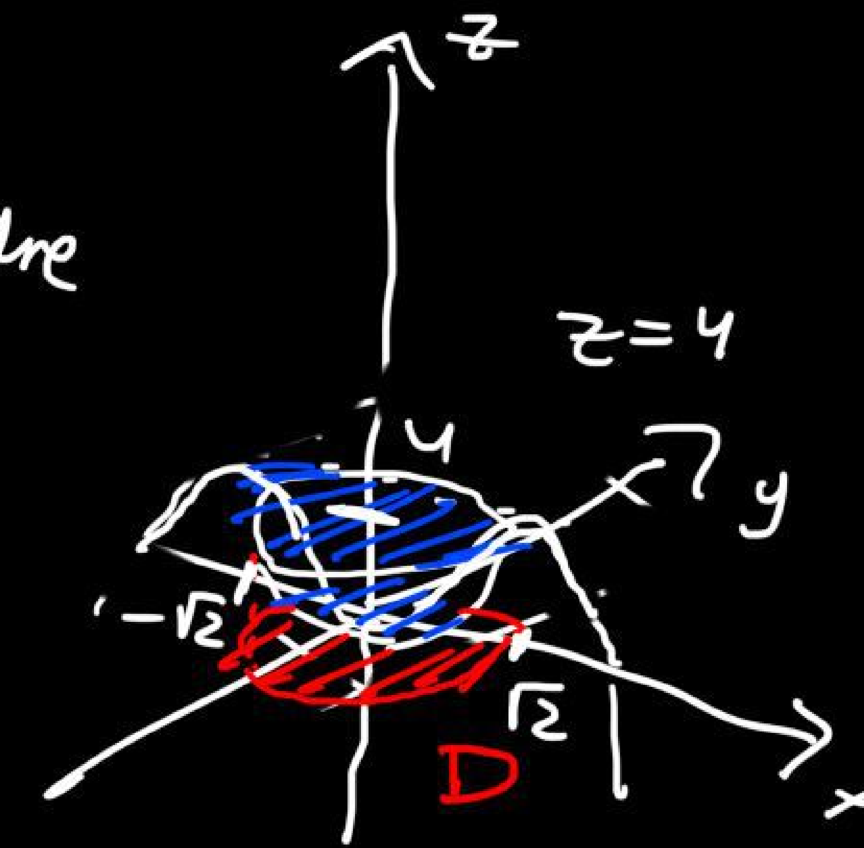
höjd skålen $\underline{\underline{4}}$

c)

Skål begränsas av

$z = f(x,y) = y^2+4x^2-x^4$ ← undre

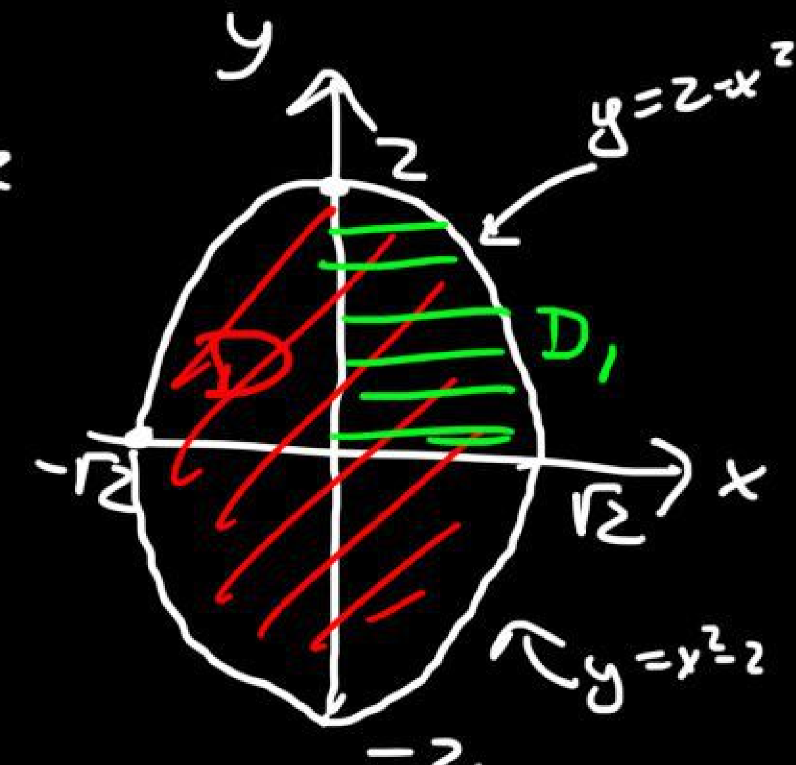
och planet $z=4$ ← övre.



$$V = \iint_D \Big(4 \underbrace{-\,(y^2+4x^2-x^4)}_{x^4-4x^2+4-y^2 = (x^2-2)^2-y^2}\Big)\,dx\,dy$$

$$= 4\iint_{D_1} \big((x^2-2)^2-y^2\big)\,dx\,dy$$

↗ symmetri.



D: $y^2+4x^2-x^4 \le 4$ Rand: $y^2+4x^2-x^4 = 4 \iff$

$-\sqrt{2}\le x\le\sqrt{2}$

$y^2 = 4-4x^2+x^4 = (x^2-2)^2$

$\iff y = \pm\sqrt{(x^2-2)^2} = \pm|x^2-2|$

$= \pm(2-x^2)$

$$4\iint_{D_1}\left((x^2-2)^2-y^2\right)dx\,dy = 4\int_0^{\sqrt{2}}\left(\int_0^{2-x^2}\left((x^2-2)^2-y^2\right)dy\right)dx =$$

$$= 4\int_0^{\sqrt{2}}\left[\underbrace{(x^2-2)^2}_{(2-x^2)^2}y-\frac{y^3}{3}\right]_0^{2-x^2}dx = 4\int_0^{\sqrt{2}}\underbrace{\left((2-x^2)^3-\frac{1}{3}(2-x^2)^3\right)}_{=\frac{2}{3}(2-x^2)^3}dx$$

$$= \frac{8}{3}\int_0^{\sqrt{2}}\underbrace{(2-x^2)^2}\,dx = \frac{8}{3}\left[8x-4x^3+\frac{6}{5}x^5-\frac{x^7}{7}\right]_0^{\sqrt{2}} =$$

$$= 1\cdot 2^3+3\cdot 2^2(-x^2)+3\cdot 2\cdot(-x^2)^2+1\cdot(-x^2)^3 =$$

$$= 8-12x^2+6x^4-x^6$$

1

1 1

1 2 1

1 3 3 1

$$= \frac{8}{3}\left(\cancel{8\sqrt{2}-8\sqrt{2}}+\frac{6}{5}\cdot 4\sqrt{2}-\frac{8}{7}\sqrt{2}\right)$$

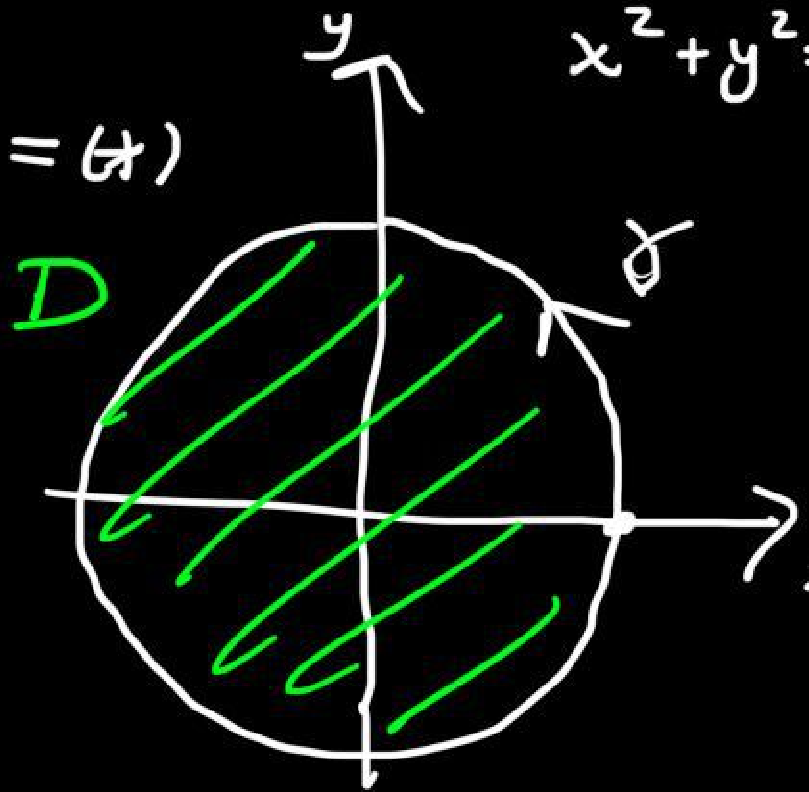
$$= \frac{8}{3}\sqrt{2}\left(\frac{24}{5}-\frac{8}{7}\right) = \frac{128\cdot 8}{3\cdot 35}\sqrt{2} = \frac{1024}{105}\sqrt{2}$$

$$\frac{168-40}{35} = \frac{128}{35}$$

9.9.

$$I = \int_\gamma \underbrace{(x^3 - x^2 y)}_{P}\,dx + \underbrace{xy^2}_{Q}\,dy = (*)$$

$x^2+y^2=2^2$



$$\boxed{(x,y) = (2\cos t, 2\sin t) \quad t: 0 \to 2\pi}$$

$$(*) = \int_0^{2\pi} (8\cos^3 t - 8\cos^2 t \sin t)(-2\sin t)\,dt + 8\cos t \sin^2 t\, 2\cos t\,dt$$

$$= \int_0^{2\pi} \left(-16\cos^3 t \sin t + 32\cos^2 t \sin^2 t\right) dt$$

$$32(\cos t \sin t)^2, \quad \underbrace{\cos t \sin t}_{\frac{1}{2}\sin 2t}$$

Jobbigt! men går

$$\frac{dx}{dt} = -2\sin t \qquad \frac{dy}{dt} = 2\cos t \quad \cdot\, dy = 2\cos t\,dt$$

$$\Leftrightarrow \; dx = -2\sin t\,dt$$

<u>Alt.</u> (Greens formel)

$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D \left(y^2 - (-x^2)\right) dx\,dy$$

$$= \iint_D (x^2+y^2)\,dx\,dy = \left[\begin{matrix} x = r\cos t & 0 \le r \le 2 \\ y = r\sin t & 0 \le \varphi \le 2\pi \end{matrix}\right]_{E:} =$$

$$= \iint_E r^2 \cdot r\,dr\,d\varphi = \left(\int_0^{2\pi} 1\,d\varphi\right)\left(\int_0^2 r^3\,dr\right) = \ldots = 8\pi$$

9.37. b) Avgör om

$$\underbrace{\frac{x-y}{x^2+y^2}}_{=P}\,dx + \underbrace{\frac{x+y}{x^2+y^2}}_{=Q}\,dy \quad \text{är } \underline{\underline{\text{exakt}}} \text{ i } \Omega : x^2+y^2>0$$

ej enkelt sammanhängande

$P\,dx+Q\,dy$ diff.formen exakt $\Leftrightarrow$ (P,Q) potentialfält/konservativt fält

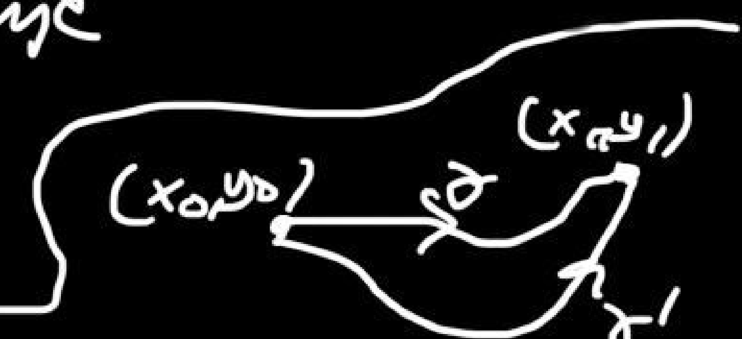
I) (P,Q) potentialfält $\overset{\text{det.}}{\Leftrightarrow}$ existerar potentialfunktion U som uppfyller $\frac{\partial U}{\partial x} = P$, $\frac{\partial U}{\partial y} = Q$

bara om Ω enk.sam. $\Leftarrow$

II) (P,Q) potentialfält $\cancel{\Leftarrow}$ $\Rightarrow$ $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

III) (P,Q) pot. fält $\Leftrightarrow$ integralen är oberoende av väg

IV) (P,Q) pot. fält $\Leftrightarrow$ $\int_\gamma P\,dx+Q\,dy = 0$ för varje sluten kurva



Sats: (P,Q) potentialfält $\Rightarrow$ $\int_\gamma P\,dx+Q\,dy = U(x_1,y_1) - U(x_0,y_0)$

slutp. startp.

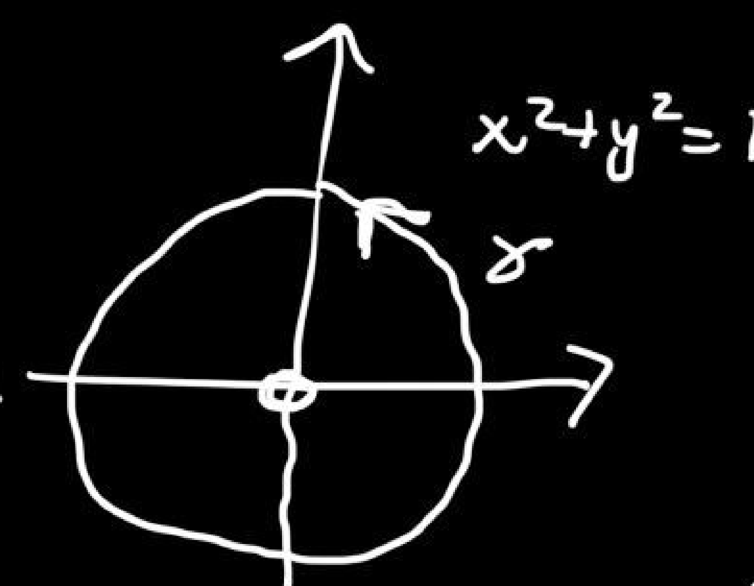
$$(P,Q) = \left(\frac{x-y}{x^2+y^2}, \frac{x+y}{x^2+y^2} \right) \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \; ?$$

- $$\frac{\partial Q}{\partial x} = \frac{1\cdot(x^2+y^2) - 2x(x+y)}{(x^2+y^2)^2} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2} \quad \text{OK}$$

$$\frac{\partial P}{\partial y} = \frac{(-1)(x^2+y^2) - 2y(x-y)}{(x^2+y^2)^2} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

(ingen slutsats kan dras)

- $(x,y) = (\cos t, \sin t) \quad t: 0 \to 2\pi$



$$\int_0^{2\pi} \frac{\cancel{\cos t} - \sin t}{1}(-\sin t)\,dt + \frac{\cos t \cancel{+\sin t}}{1}\cos t\,dt$$

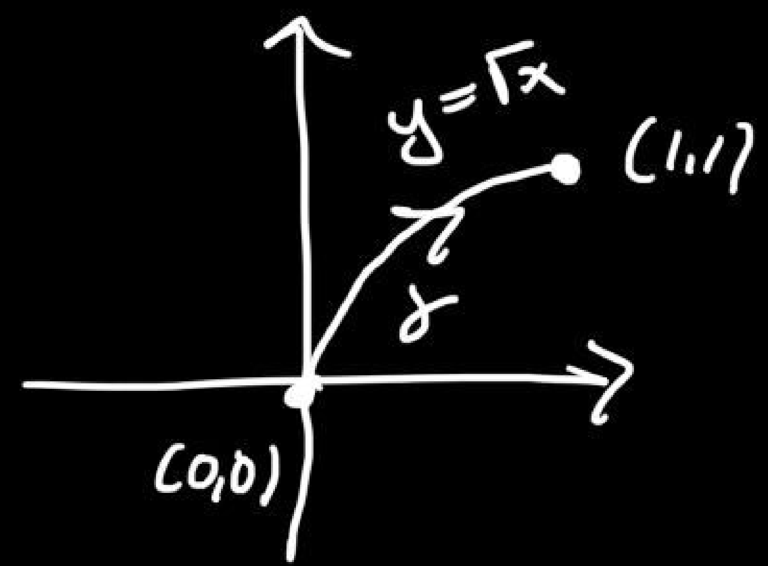
$$= \int_0^{2\pi} (\underbrace{\cos^2 t + \sin^2 t}_{=1})\,dt = \int_0^{2\pi} 1\,dt = 2\pi \neq 0 \,!$$

$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t$$

Slutsats <u>ej</u> potentialfält.

9.38. Beräkna $$\int_\gamma \underbrace{\frac{y^2}{1+x^2y^2}}_{P}\,dx + \underbrace{\left(\frac{xy}{1+x^2y^2} + \arctan(xy)\right)}_{Q}dy$$

i $\underline{\underline{\Omega = \mathbb{R}^2}}$



Parametrisering verkar hopplöst!

enkelt sammanh.!

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \; ? \quad \text{Kolla!}$$

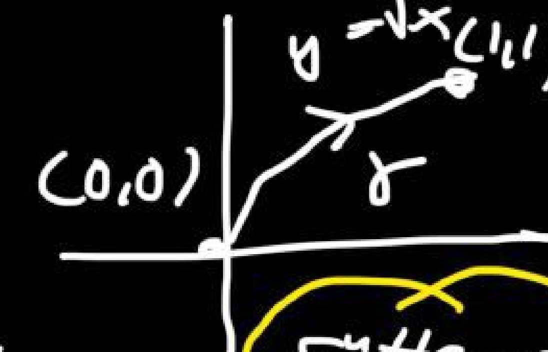
$$\frac{\partial Q}{\partial x} = \frac{y(1+x^2y^2) - 2xy^2\cdot xy}{(1+x^2y^2)^2} + \frac{y}{\underbrace{(xy)^2+1}_{1+x^2y^2}} =$$

$$= \frac{y + \cancel{x^2y^3} - \cancel{2x^2y^3} + y(1+\cancel{x^2y^2})}{(1+x^2y^2)^2} = \frac{2y}{(1+x^2y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{2y(1+\cancel{x^2y^2}) - \cancel{2x^2y}\cdot y^2}{(1+x^2y^2)^2} = \frac{2y}{(1+x^2y^2)^2} \quad \text{OK!}$$

Ω enkelt sammanhängande + $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ $\Rightarrow$ (P,Q) $\underline{\underline{\text{potentialfält}}}$

$$(P,Q) = \left(\frac{y^2}{1+x^2y^2}, \frac{xy}{1+x^2y^2}+\arctan(xy)\right)$$



$$\begin{cases} \text{I)} & \dfrac{\partial U}{\partial x} = \dfrac{y^2}{1+x^2y^2} \\ \text{II)} & \dfrac{\partial U}{\partial y} = \dfrac{xy}{1+x^2y^2} + \arctan(xy) \end{cases}$$

$$\text{I)} \quad U = \int \frac{y^2}{\underbrace{1+x^2y^2}_{1+(xy)^2}}\,dx = y\cdot\arctan(xy) + \varphi(y)$$

jämför med

$$\text{II)}: \quad \frac{\partial U}{\partial y} = 1\cdot\arctan(xy) + y\,\frac{x}{1+x^2y^2} + \varphi'(y)$$

$$=$$

Slutsats. $\varphi'(y) = 0 \Leftrightarrow \varphi(y) = C$

Potentialfunktionen $U(x,y) = y\arctan(xy) + C$

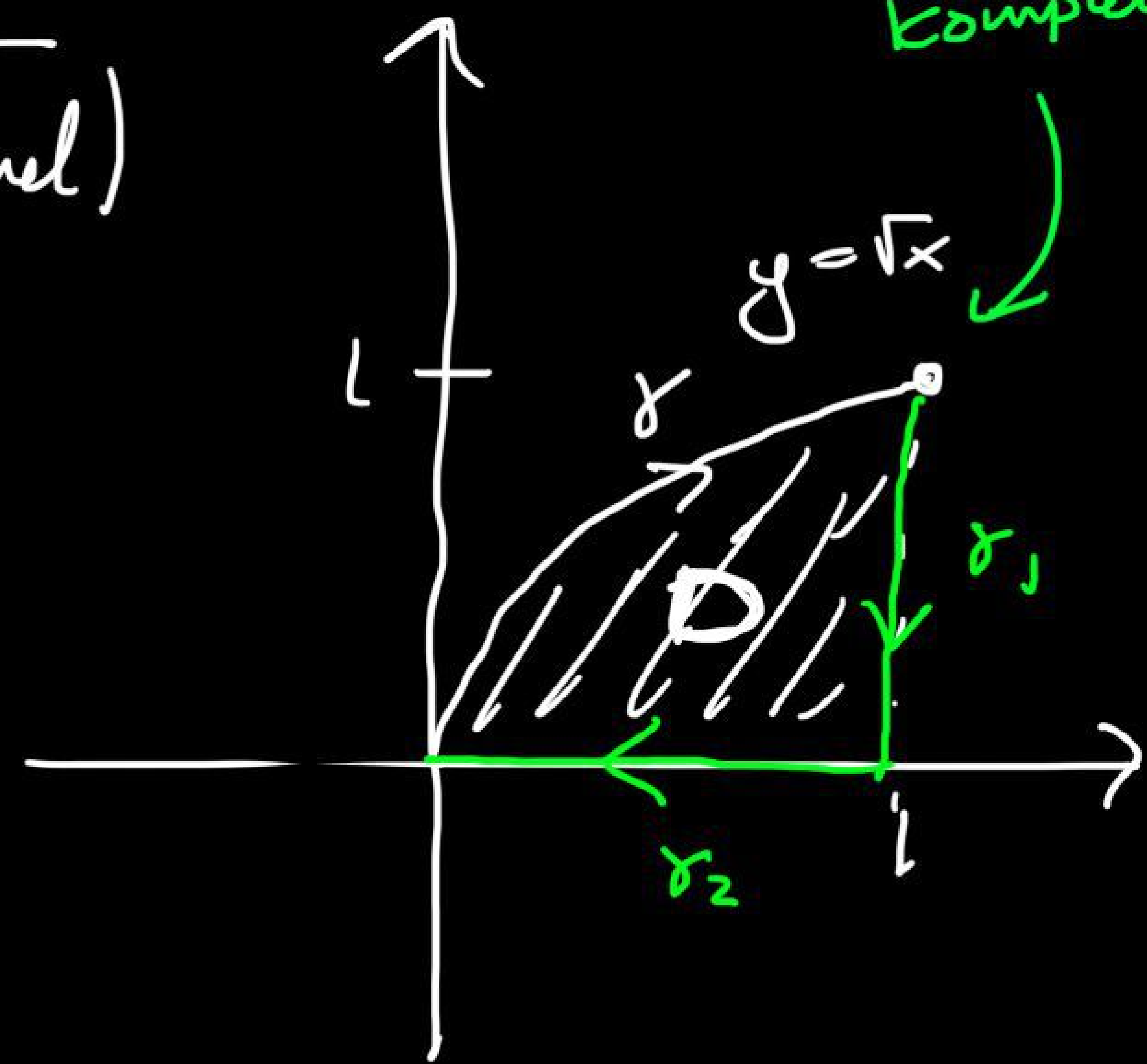
Sätt $C=0$

Integralen blir nu

$$I = U(1,1) - U(0,0) =$$
$$= 1\cdot\arctan 1 - 0\cdot\arctan 0$$
$$= \arctan 1 = \frac{\pi}{4}$$

Alt. lösning

(Greens formel)



komplettera!

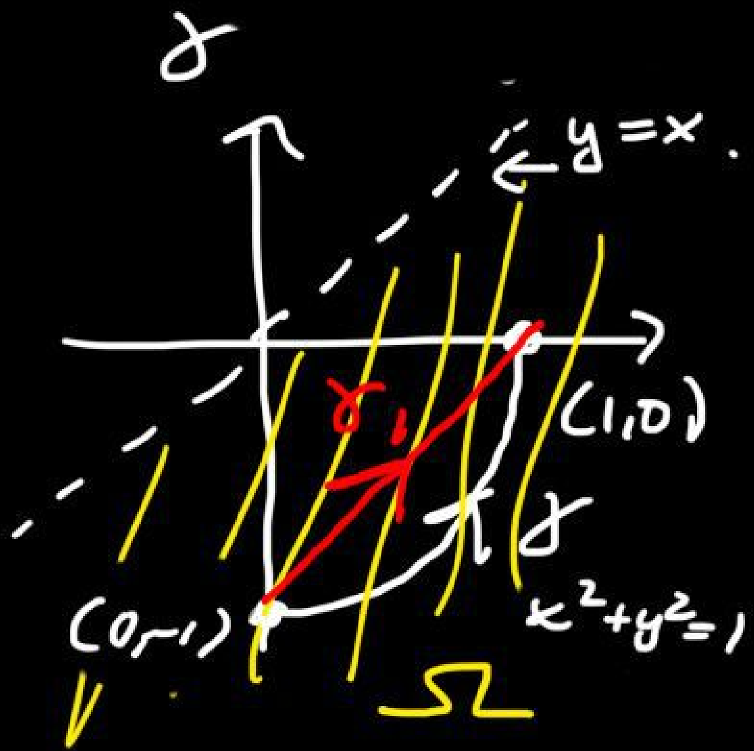
Slutför som övning!

OBS!

$$\int_{\gamma+\gamma_1+\gamma_2} P\,dx + Q\,dy = -\iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dx\,dy$$

9.46. $I=\int_\gamma \frac{-2y}{(x-y)^3}\,dx+\frac{x+y}{(x-y)^3}\,dy$ $\qquad y\neq x$

$P=$ $\qquad Q=$



$$\frac{\partial Q}{\partial x}=\ldots\ldots=\frac{-2x-4y}{(x-y)^4}$$

$$\frac{\partial P}{\partial y}=\ldots\ldots=\frac{-2x-4y}{(x-y)^4}\qquad \text{ok!}$$

Området $\Omega: y<x$ är enkelt sammanhängande +

$+\ \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ i Ω $\Rightarrow$ (P,Q) är ett potentialfält i Ω

$\Leftrightarrow$ integralen är ober. av vägen

$$I=\int_{\gamma_1}P\,dx+Q\,dy=\int_0^1\frac{-2\cancel{(t}-1)}{1^3}\,dt+\frac{\cancel{2t}-1}{1^3}\,dt=$$

$\boxed{\gamma_1: (x,y)=(t,t-1),\ t:0\to 1}$ $\qquad =\int_0^1 1\,dt=\underline{\underline{1}}$