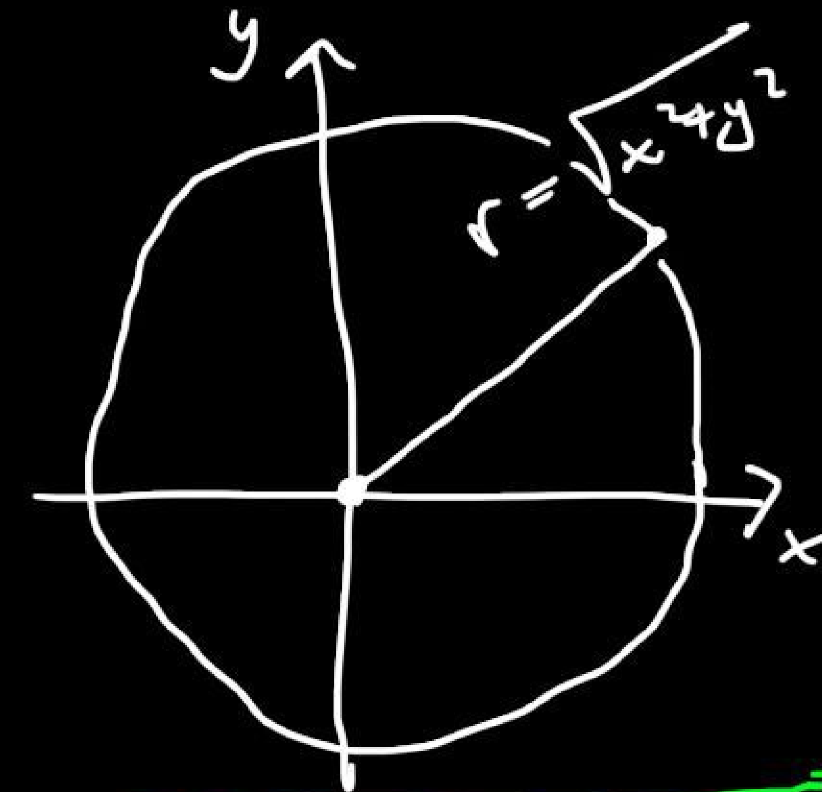
4.57. $T = T(r,t)$

$$\boxed{T'_t = T''_{rr} - \frac{1}{r}T'_r}$$



Sök $T(r,t) = f\left(\frac{r}{\sqrt{t}}\right)$. Bestäm alla sådana f.

lösn.

$$T'_t = f'\left(\frac{r}{\sqrt{t}}\right)\cdot\left(-\frac{1}{2}\right)\frac{r}{t^{3/2}}$$

$$\boxed{f''(s) + \left(\frac{1}{2}s - \frac{1}{s}\right)f'(s) = 0}$$

$$T'_r = f'\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{\sqrt{t}}$$

$$T''_{rr} = f''\left(\frac{r}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}\cdot\frac{1}{\sqrt{t}} = f''\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{t}$$

$$f'\left(\frac{r}{\sqrt{t}}\right)\left(-\frac{1}{2}\right)\frac{r}{t^{3/2}} = f''\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{t} - \frac{1}{r\sqrt{t}}f'\left(\frac{r}{\sqrt{t}}\right)$$

$$\Leftrightarrow f''\left(\frac{r}{\sqrt{t}}\right)\cdot\frac{1}{t} + \left(\frac{1}{2}\frac{r}{t^{3/2}} - \frac{1}{r\sqrt{t}}\right)f'\left(\frac{r}{\sqrt{t}}\right) = 0$$

$s = \frac{r}{\sqrt{t}}$

$$\Leftrightarrow f''\left(\frac{r}{\sqrt{t}}\right) + \left(\frac{1}{2}\frac{r}{\sqrt{t}} - \frac{\sqrt{t}}{r}\right)f'\left(\frac{r}{\sqrt{t}}\right) = 0$$

$$f''(s) + \left(\tfrac{1}{2}s - \tfrac{1}{s}\right) f'(s) = 0$$

$$\boxed{h(s) = f'(s)}$$

$$h'(s) + \underbrace{\left(\tfrac{1}{2}s - \tfrac{1}{s}\right)}_{g(s)} h(s) = 0$$

$$g(s) = \tfrac{1}{2}s - \tfrac{1}{s} \qquad s>0$$

$$G(s) = \tfrac{1}{4}s^2 - \ln s$$

Int. f. $e^{G(s)} = e^{\frac{1}{4}s^2 - \ln s} =$

$$= e^{\frac{1}{4}s^2} \cdot e^{-\ln s}$$

$$= \frac{e^{\frac{1}{4}s^2}}{s}$$

$$\frac{e^{\frac{1}{4}s^2}}{s} h'(s) + \frac{e^{\frac{1}{4}s^2}}{s}\left(\tfrac{1}{2}s - \tfrac{1}{s}\right) h(s) = 0$$

$$\left(\frac{e^{\frac{1}{4}s^2}}{s} \cdot h(s)\right)' = 0$$

$$\Leftrightarrow \frac{e^{\frac{1}{4}s^2}}{s} h(s) = \int 0 \, ds = C$$

$$\Leftrightarrow h(s) = \frac{Cs}{e^{\frac{1}{4}s^2}} = Cse^{-\frac{1}{4}s^2}$$

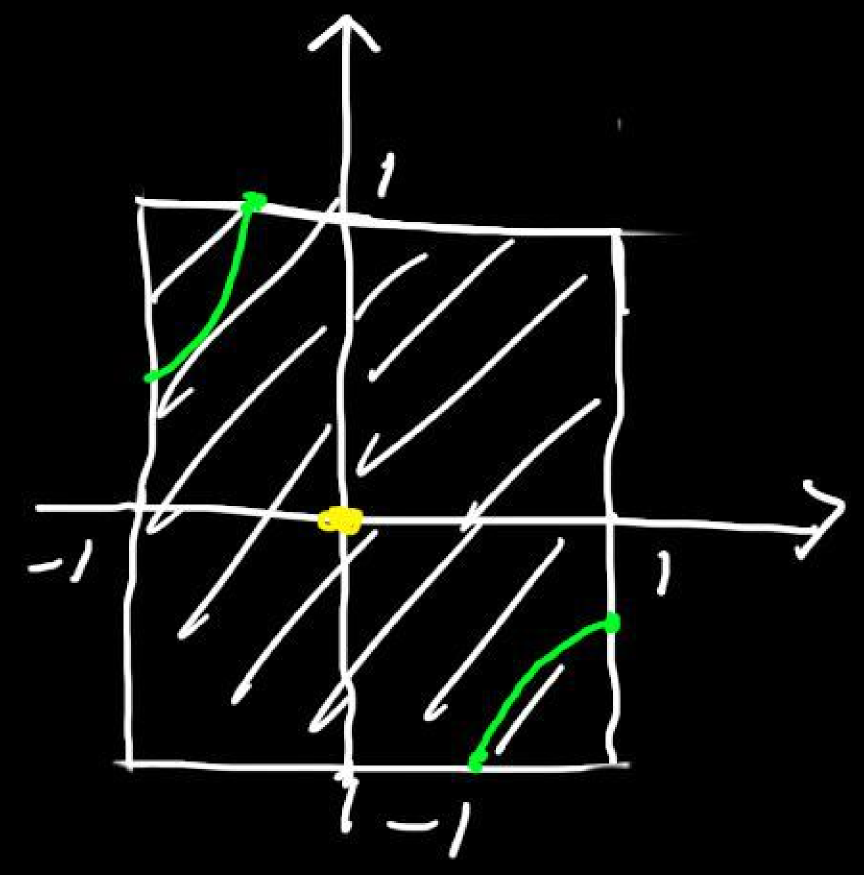
$$f'(s) = Cse^{-\frac{1}{4}s^2} \Leftrightarrow \underline{f(s)} = \int Cse^{-\frac{1}{4}s^2} ds = \overbrace{(-2)C}^{E} e^{-\frac{1}{4}s^2} + D$$

$$\underline{= Ee^{-\frac{1}{4}s^2} + D}$$

5.17. $f(x,y) = xy + x^2y^2$

$|x| \leq 1$, $|y| \leq 1$

$-1 \leq x \leq 1$ $-1 \leq y \leq 1$



f kont., kompakt område (sluten, begränsad)

$\Rightarrow f_{max}, f_{min}$ existerar

I/ stat. punkter
II) randen
III) hörnen.

I/ $\begin{cases} f'_x = y + 2xy^2 = 0 \\ f'_y = x + 2x^2y = 0 \end{cases}$

$\Leftrightarrow \begin{cases} y(1+2xy) = 0 & a) \\ x(1+2xy) = 0 & b) \end{cases}$

a) $\underline{y=0}$ insatt i b) ger $x=0$

$f(0,0) = \boxed{0}$

$\underline{1+2xy=0}$ — ‖ — ger $0=0$

$1+2xy=0 \Leftrightarrow y = -\frac{1}{2x}$

$f(x,-\frac{1}{2x}) = x(-\frac{1}{2x}) + x^2(-\frac{1}{2x})^2$
$= -\frac{1}{2} + \frac{1}{4} = \boxed{-\frac{1}{4}}$

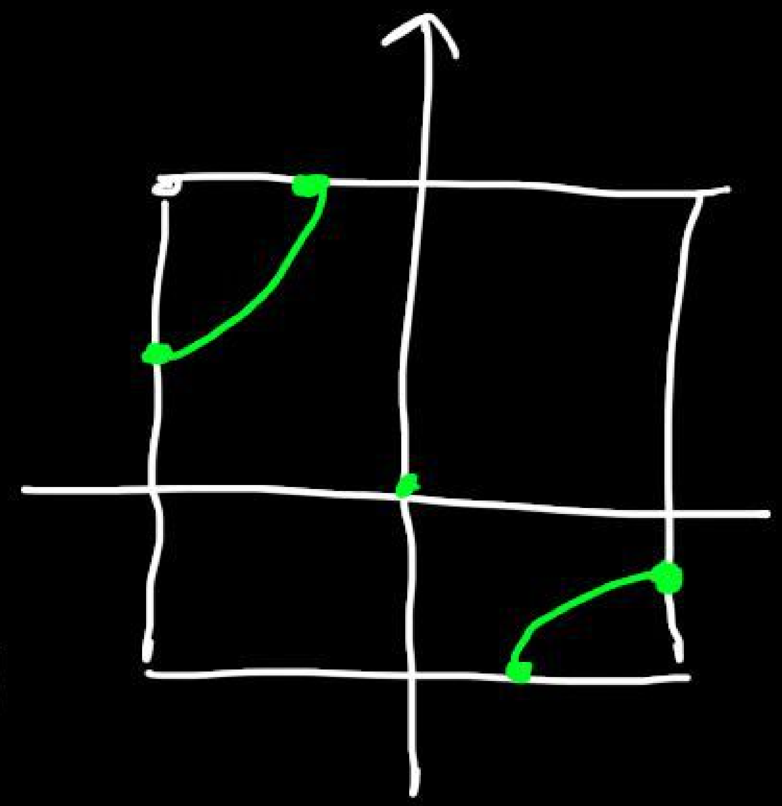
$\leftarrow$ alla punkter som uppfyller $1+2xy=0$ inuti området är stationära

II)

$\underline{y=1}$ $g_1(x) = f(x,1) = x + x^2$, $-1 \le x \le 1$

$g_1'(x) = 1 + 2x = 0 \iff x = -\frac{1}{2}$

$f(-\frac{1}{2}, 1) = \boxed{-\frac{1}{4}}$



$\underline{x=1}$ $g_2(y) = f(1,y) = y + y^2$, $-1 \le y \le 1$

$g_2'(y) = 1 + 2y = 0 \iff y = -\frac{1}{2}$

$f(1, -\frac{1}{2}) = \boxed{-\frac{1}{4}}$

$\underline{y=-1}$: . . . . $f(\frac{1}{2}, -1) = \boxed{-\frac{1}{4}}$

$\underline{x=-1}$: . . . . $f(-1, 1/2) = \boxed{-\frac{1}{4}}$

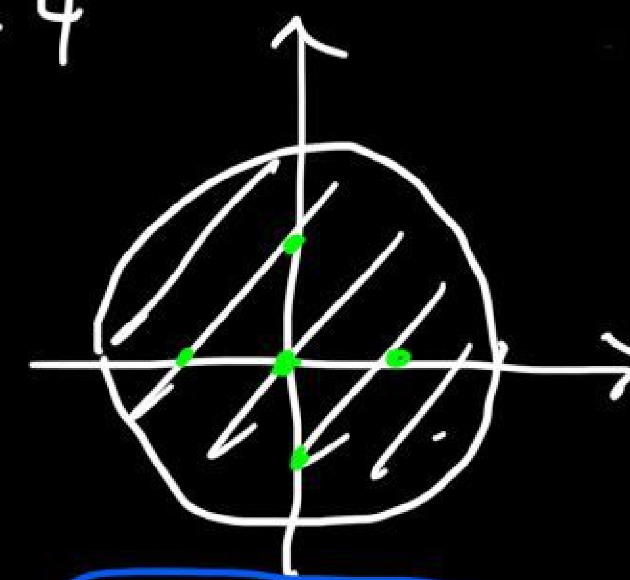
$\underline{III)}$ $f(1,1) = f(-1,-1) = \boxed{2}$

$f(1,-1) = f(-1,1) = \boxed{0}$

$\underline{\text{Svar:}}$ $f_{min} = -\frac{1}{4}$

$f_{max} = 2$

5.26 $f(x,y) = (x^2+3y^2)e^{-x^2-y^2}$, $x^2+y^2 \le 4$



f kont. på kompakt

I) $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$ ....

$f(0,0) = \boxed{0}$

$f(0,\pm 1) = \boxed{3e^{-1}}$

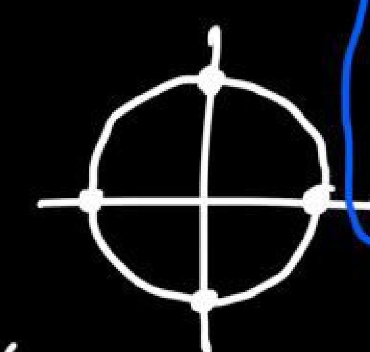
$f(\pm 1,0) = \boxed{e^{-1}}$

$\frac{3}{e} > \frac{12}{e^4} \iff e^3 > 4$ ok

II) $\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$ $0 \le t \le 2\pi$

$g(t) = f(2\cos t, 2\sin t) = (4\cos^2 t + 12\sin^2 t)e^{-4\overbrace{(\cos^2 t + \sin^2 t)}^{=1}}$

$= (\underbrace{4\cos^2 t + 4\sin^2 t}_{=4} + 8\sin^2 t)e^{-4}$



$g'(t) = 8e^{-4} \cdot 2\sin t\cos t = 0 \iff \sin t = 0$ eller $\cos t = 0$

inre punkter $t = \pi/2, \pi, 3\pi/2$

III) $f(2,0) = \boxed{4e^{-4}}$, $f(0,2) = \boxed{12e^{-4}}$, $f(-2,0) = \boxed{4e^{-4}}$

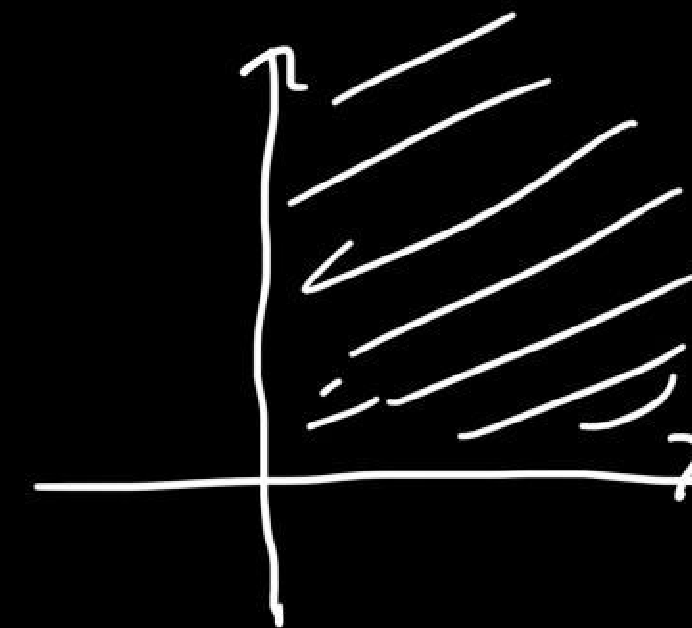
$f(0,-2) = \boxed{12e^{-4}}$

fmin 0, $3e^{-1}$ = fmax, e^{-1}, $4e^{-4}$, $12e^{-4}$

5.33. $f(x,y) = (x^2+y)e^{-x-y}$

$0 \le x < \infty$

$0 \le y < \infty$



omr. <u>ej</u> kompakt

Studera $f(x,y)$ då $r=\sqrt{x^2+y^2} \to \infty$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad f(x,y) = \frac{r^2\cos^2\varphi + r\sin\varphi}{e^{r(\cos\varphi+\sin\varphi)}}$$

$$\underset{\downarrow\ 0}{0} \le \Big|\overset{\nearrow 0}{f(x,y)}\Big| = \frac{|r^2\cos^2\varphi + r\sin\varphi|}{e^{r(\cos\varphi+\sin\varphi)}} \overset{*)}{\le} \frac{r^2\overset{\le 1}{|\cos^2\varphi|} + r\overset{\le 1}{|\sin\varphi|}}{e^{r\cdot 1}} \le$$

$$\le \underset{0\swarrow}{\frac{r^2}{e^r}} + \underset{\swarrow 0}{\frac{r}{e^r}} \to 0$$

*) $\cos\varphi + \sin\varphi = 1\cdot\cos\varphi + 1\cdot\sin\varphi =$

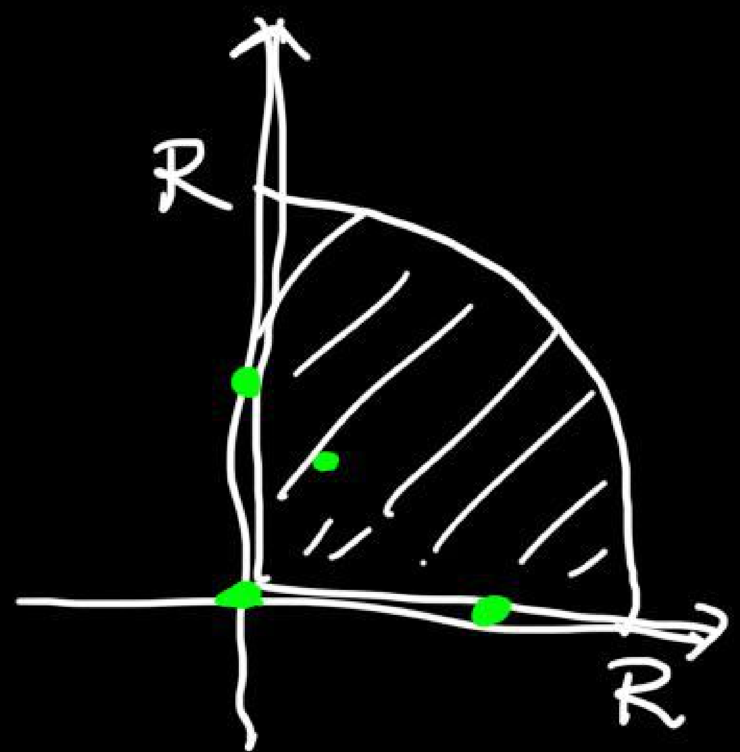
$$= \underset{\sqrt{2}}{\underbrace{\sqrt{1^2+1^2}}}\left(\frac{1}{\sqrt{2}}\cos\varphi + \frac{1}{\sqrt{2}}\sin\varphi\right) = \underbrace{\sqrt{2}\sin\left(\varphi+\frac{\pi}{4}\right)}_{1 \le \quad \le \sqrt{2}}$$

$0 \le \varphi \le \pi/2$

$\frac{\pi}{4} \le \varphi + \pi/4 \le \frac{3\pi}{4}$

$$\underbrace{\sqrt{2}\,\underbrace{\sin\left(\varphi+\frac{\pi}{4}\right)}_{\frac{1}{\sqrt{2}} \le \quad \le 1}}_{1 \le \quad \le \sqrt{2}}$$

$$\begin{cases} f'_x = \cancel{e^{-x-y}}\,(2x - x^2 - y) = 0 & a) \\ f'_y = \cancel{e^{-x-y}}\,(1 - x^2 - y) = 0 & b) \end{cases}$$



a) $y = 2x - x^2$ sätt in i b)!

$1 - x^2 - (2x - x^2) = 0 \iff 1 - 2x = 0$

$\iff x = 1/2$

ger $y = 2 \cdot \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$f\left(\frac{1}{2}, \frac{3}{4}\right) = \boxed{e^{-5/4}}$

$g_1(x) = f(x,0) = x^2 e^{-x}, \quad x \geq 0$

$\Rightarrow g_1'(x) = 2x e^{-x} - x^2 e^{-x} = e^{-x} x(2 - x) = 0 \iff x = 0$ eller $x = 2$

$f(0,0) = \boxed{0}, \quad f(2,0) = \boxed{4e^{-2}}$

$g_2(y) = f(0,y) = \ldots\ldots\ldots$ $\quad f(0,1) = \boxed{e^{-1}}$

$$\frac{4}{e^2} = \frac{4}{e} \cdot \frac{1}{e} > \frac{1}{e} > \frac{1}{e^{5/4}}$$

Formellt: $f(x,y) \geq 0 \Rightarrow f_{min} = 0$

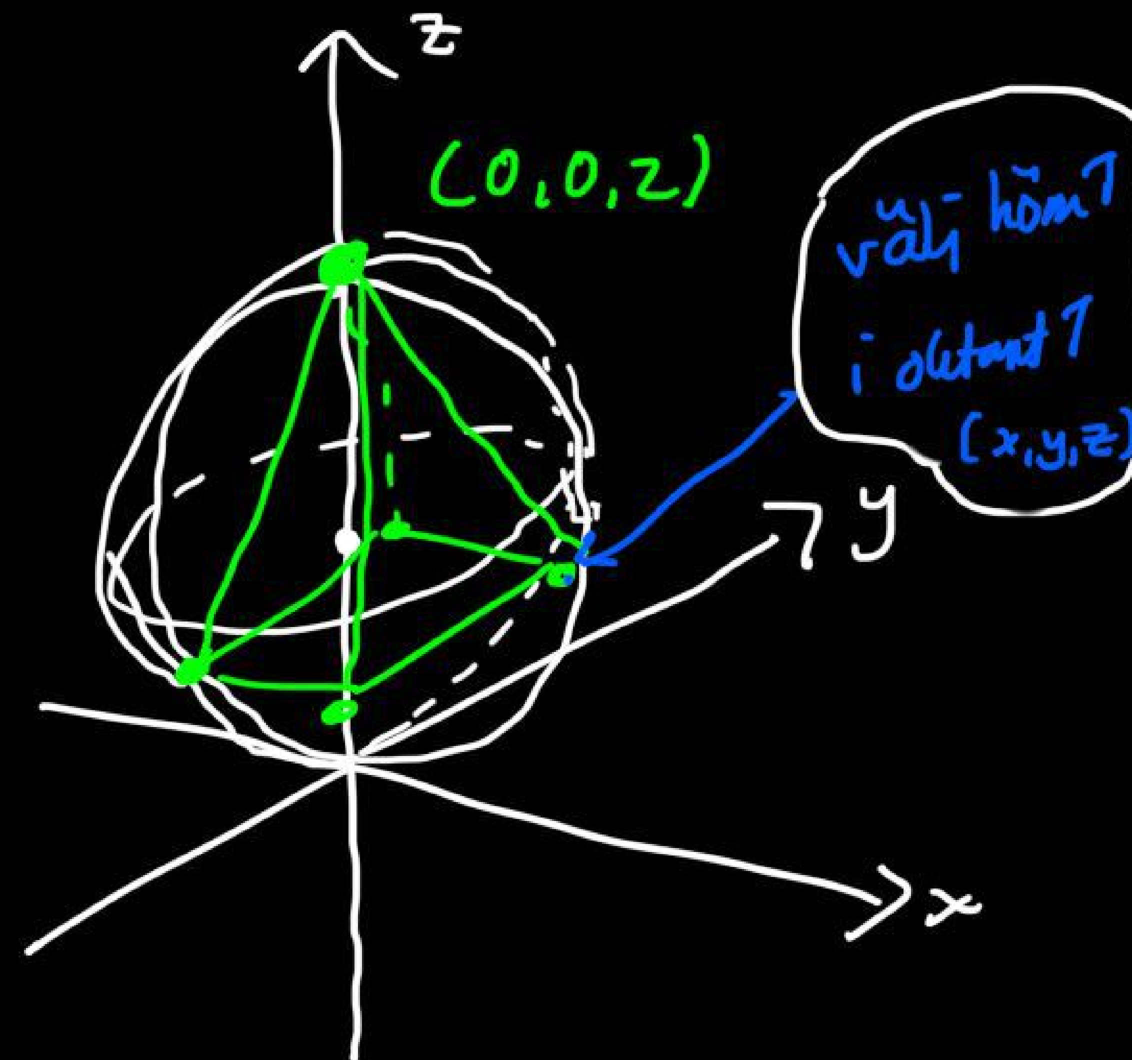
$0 \leq f(x,y) \leq 4e^{-2}$ då $\sqrt{x^2 + y^2} \geq R$ för något R

slutsats $f_{max} = 4e^{-2}$

5.48. $x^2 + \frac{y^2}{4} + (z-1)^2 = 1$

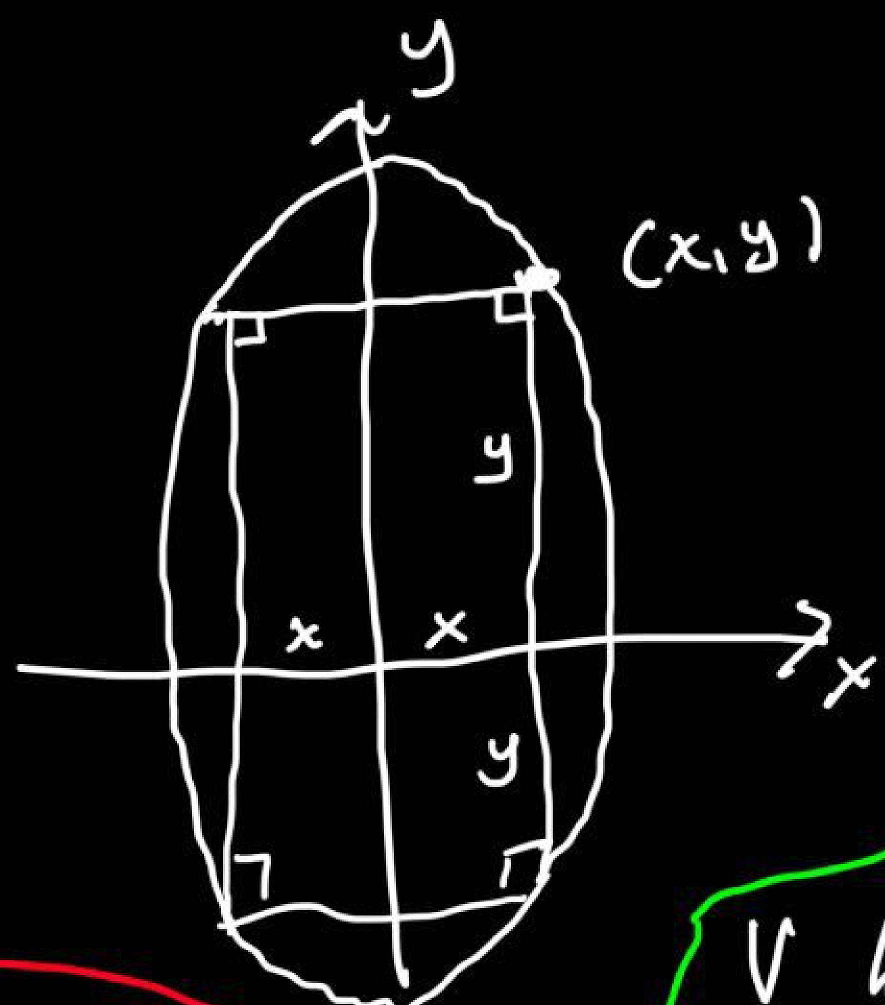
Maximera volymen av pyramiden!

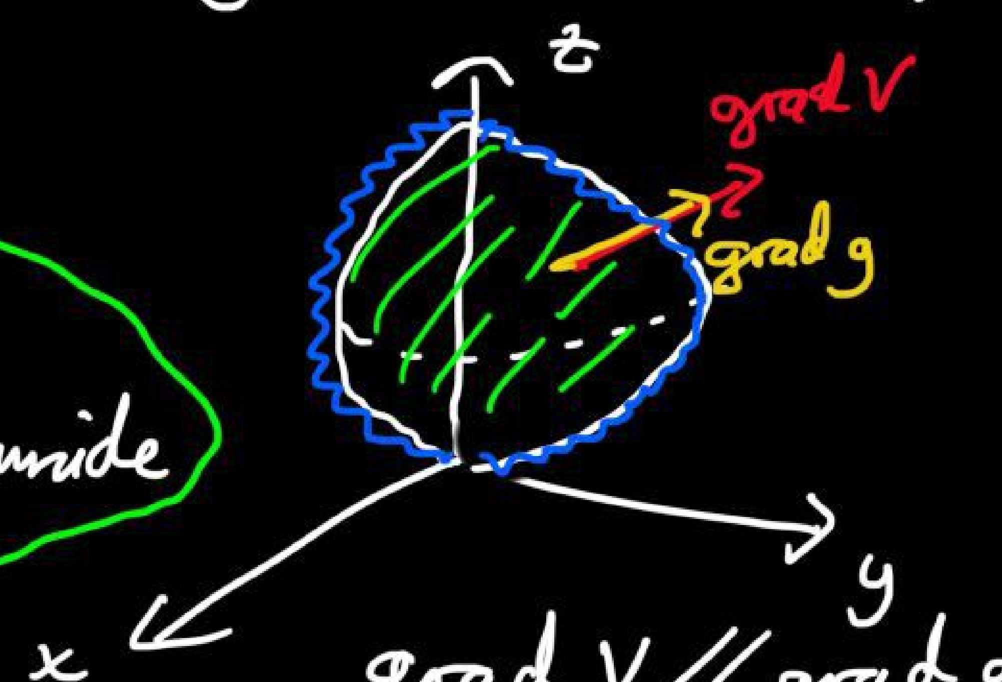
$$V(x,y,z) = \frac{B \cdot h}{3} = \frac{4xy(2-z)}{3}, \quad x,y,z \geq 0$$



+ bivillkor

$$g(x,y,z) = x^2 + \frac{y^2}{4} + (z-1)^2 = 1$$





$g(x,y,z) = 1$

$x,y,z \geq 0$

grad V // grad g

V kont på kompakt område

fmax och fmin exist.

II) randen

$V(x,y,z) = \boxed{0}$

$$V(x,y,z) = \frac{4xy(2-z)}{3}$$

$$g(x,y,z) = x^2 + \frac{y^2}{4} + (z-1)^2 = 1$$

II) grad V // grad g

$\Leftrightarrow$

grad $V = \lambda$ grad g

eller

grad $g = \bar{0}$

grad $g = \bar{0}$ $\Leftrightarrow$ $(x,y,z) = (0,0,1)$ ej på ellipsoid

$$\text{grad } V = \left(\frac{4}{3}y(2-z), \frac{4}{3}x(2-z), -\frac{4}{3}xy\right)$$

$$\text{grad } g = \left(2x, y/2, 2(z-1)\right)$$

$$\begin{cases} \frac{4}{3}y(2-z) = \lambda 2x \\ \frac{4}{3}x(2-z) = \lambda \frac{y}{2} \\ -\frac{4}{3}xy = \lambda 2(z-1) \end{cases} \Leftrightarrow \begin{cases} \lambda = \frac{2}{3}\frac{y}{x}(2-z) \\ \lambda = \frac{8}{3}\frac{x}{y}(2-z) \\ \lambda = -\frac{2}{3}\frac{xy}{z-1} \end{cases}$$

1) 2)

1) $\frac{2}{3}\frac{y}{x}(2\cancel{-z}) = \frac{8}{3}\frac{x}{y}(2\cancel{-z}) \Leftrightarrow y^2 = 4x^2 \Leftrightarrow x^2 = \frac{y^2}{4}$

2) $\frac{8}{3}\frac{x}{y}(2-z) = -\frac{2}{3}\frac{xy}{z-1} \Leftrightarrow \ldots \Leftrightarrow y^2 = 4(2-z)(1-z)$

$$V\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right) = \boxed{\frac{128}{81}}$$

$$(2-z)(1-z) + (2-z)(1-z) + (z-1)^2 = 1$$

$\Leftrightarrow \ldots\ldots \Leftrightarrow z^2 - \frac{8}{3}z + \frac{4}{3} = 0 \Leftrightarrow (z = 2)$ eller $z = 2/3$

$z = 2/3$ ger $y = 4/3$ och $x = 2/3$ Intr. punkt $(2/3, 4/3, 2/3)$