4.34. a) $f(x,y) = xy$

Nivåkurvor, gradient

$\boxed{f(x,y) = C}$

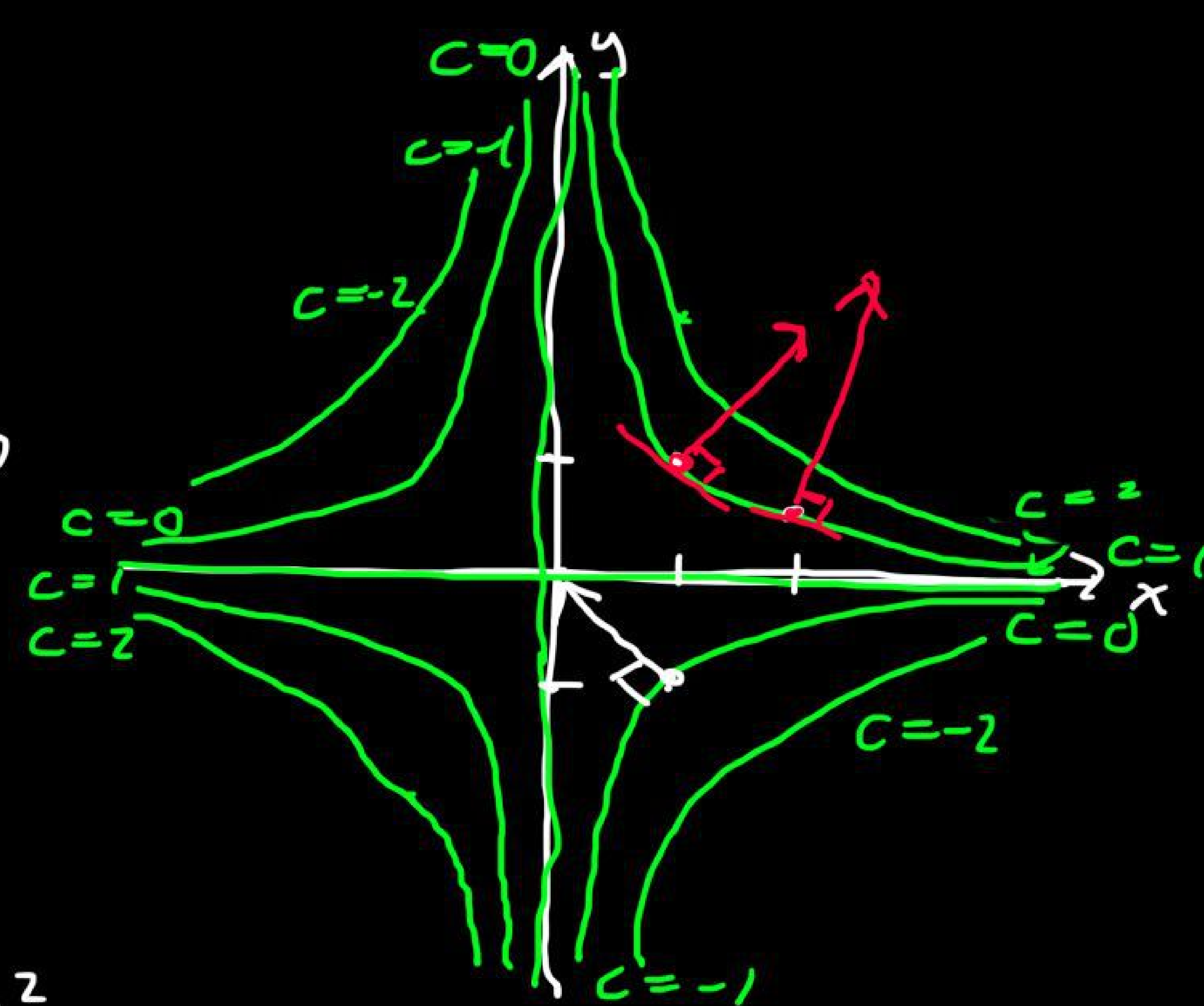
$xy = 0 \iff x = 0$ eller $y = 0$

$xy = 1 \iff y = \frac{1}{x}$

$xy = 2 \iff y = \frac{2}{x}$

$xy = -1 \iff y = -\frac{1}{x}$

$xy = -2 \iff y = -\frac{2}{x}$



$\operatorname{grad} f = (f'_x, f'_y) = (y, x)$

$(\operatorname{grad} f)(1,1) = (1,1)$

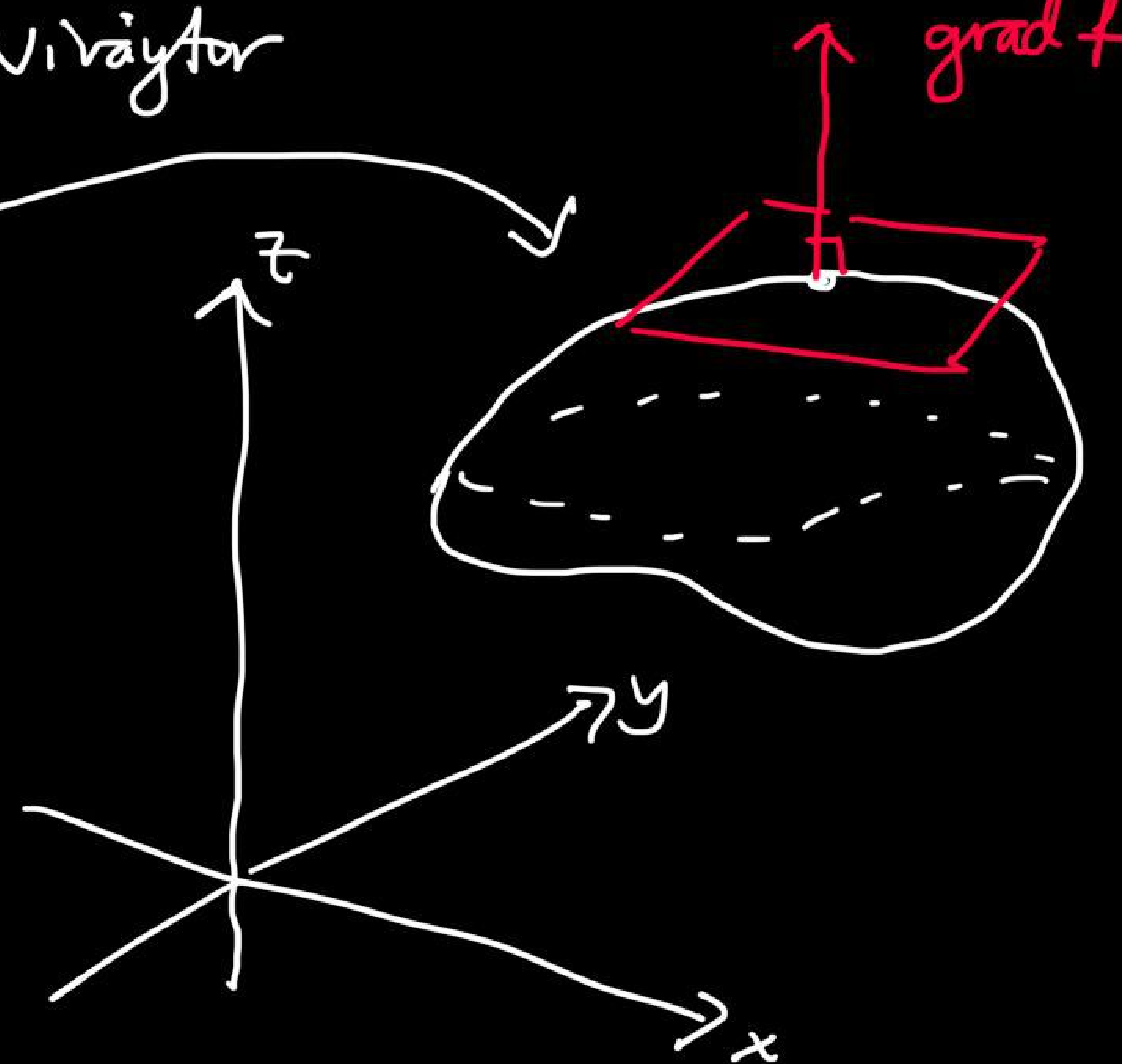
$(\operatorname{grad} f)(2,\frac{1}{2}) = (\frac{1}{2}, 2)$

$(\operatorname{grad} f)(1,-1) = (-1,1)$

- grad f $\perp$ nivåkurvan
- grad f pekar i den riktning där det är brantast uppåt
- $|\operatorname{grad} f| = \sqrt{2}$

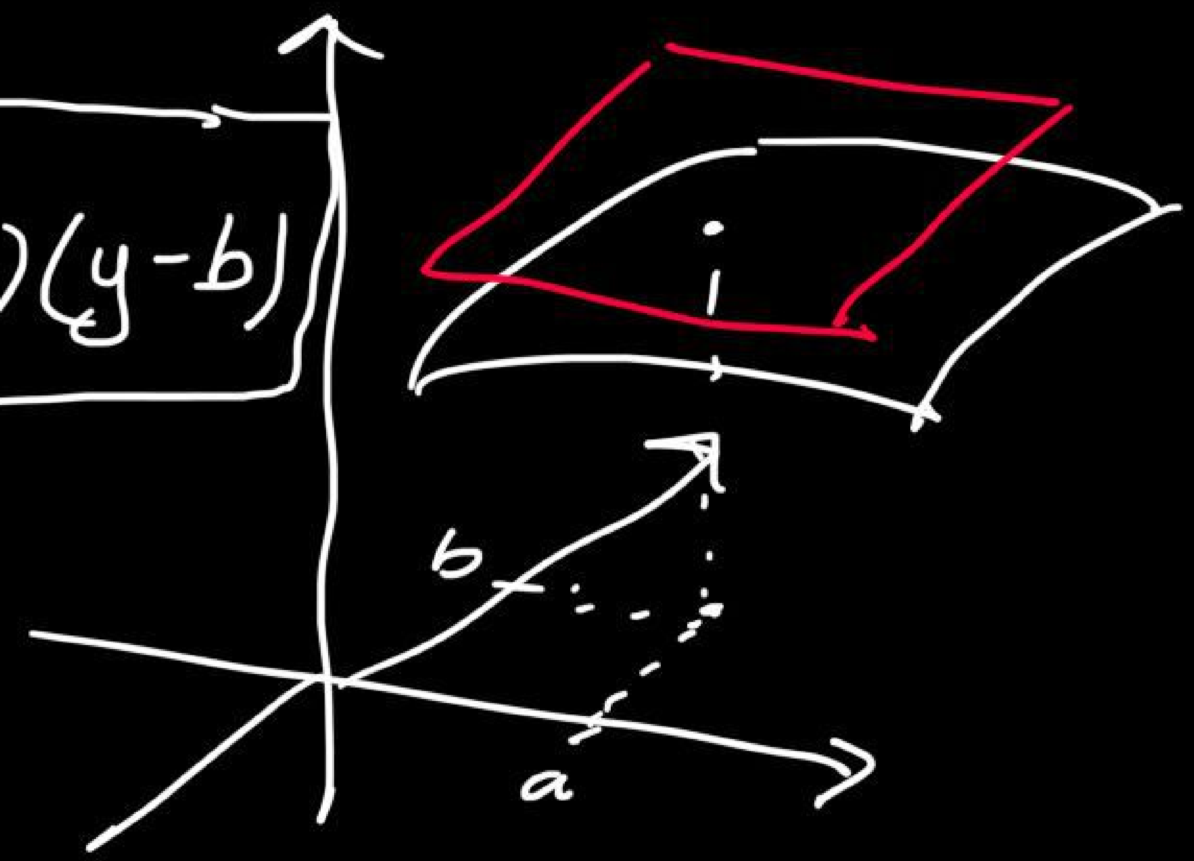
$f(x,y,z) = C$ Niváytor

$\operatorname{grad} f = (f'_x, f'_y, f'_z)$



b) $z = f(x,y)$

$$\boxed{z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)}$$



$z = f(x,y) = x^2y^3$

i punkten $(2,1,4)$

$f'_x = 2xy^3$, $f'_y = 3x^2y^2$

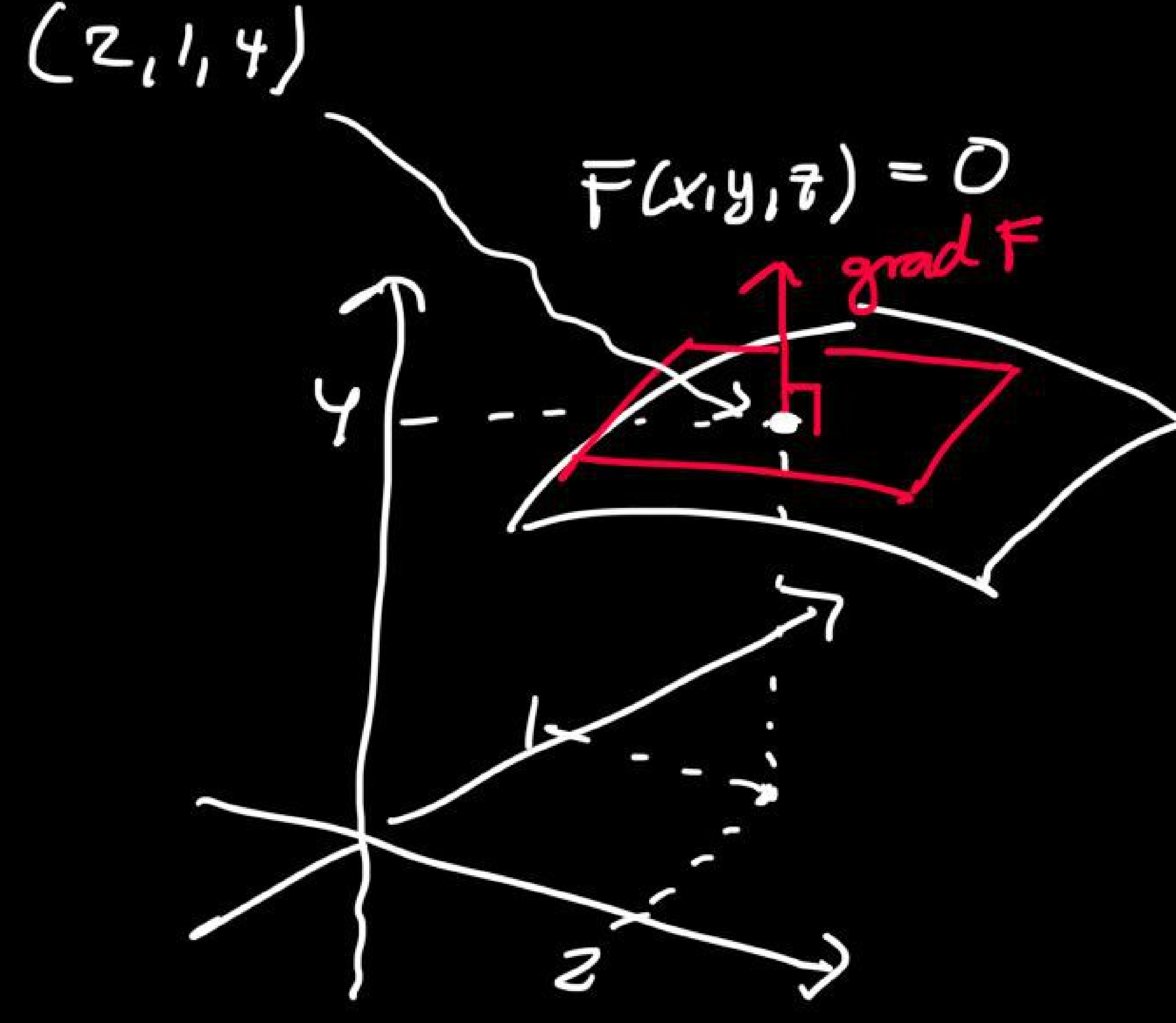
$f(2,1) = 4$, $f'_x(2,1) = 4$, $f'_y(2,1) = 12$

$z = 4 + 4(x-2) + 12(y-1)$

$\Leftrightarrow \quad \underline{4x + 12y - z - 16 = 0}$

alt. $z = f(x,y) = x^2y^3$

$\iff F(x,y,z) = x^2y^3 - z = 0$



$\operatorname{grad} F = (2xy^3, 3x^2y^2, -1)$

$(\operatorname{grad} F)(2,1,4) = (4, 12, -1)$

$Ax + By + Cz + D = 0$

$4x + 12y - z + D = 0$ Sätt in punkten

$4 \cdot 2 + 12 \cdot 1 - 4 + D = 0 \iff D = -16$

$4x + 12y - z - 16 = 0$

4.53. a) $y f'_x(x,y) - x f'_y(x,y) = xy f(x,y)$ svår!

$$\begin{cases} u = x^2 + y^2 \\ v = e^{-x^2/2} \end{cases}$$ $\underset{\sim}{x, y > 0}$ variabelbyte

$$y\left(\cancel{\tilde{f}'_u \cdot 2x} + \tilde{f}'_v(-x e^{-x^2/2})\right) \cancel{-2xy \tilde{f}'_u} = xy \tilde{f}$$

$$-\cancel{xy} \underbrace{e^{-x^2/2}}_{= v} \tilde{f}'_v = \cancel{xy} \tilde{f}$$

$$v \tilde{f}'_v + \tilde{f} = 0 \iff \tilde{f}'_v + \frac{1}{v} \tilde{f} = 0 \iff v \tilde{f}'_v + \tilde{f} = 0 \iff (\tilde{f} \cdot v)'_v = 0$$

$$\iff \tilde{f} \cdot v = \int 0 \, dv = \varphi(u)$$

$$\tilde{f}(u,v) = \frac{\varphi(u)}{v}$$

godt. derivata

$g(v) = \frac{1}{v}$

$G(v) = \ln v$

Int. faktor $e^{G(v)} = e^{\ln v} = v$

$$\boxed{f(x,y) = \tilde{f}(u(x,y), v(x,y))}$$

$$f'_x = \tilde{f}'_u \cdot u'_x + \tilde{f}'_v \cdot v'_x =$$
$$= \tilde{f}'_u \cdot 2x + \tilde{f}'_v \cdot (-x e^{-x^2/2})$$

$$f'_y = \tilde{f}'_u \cdot u'_y + \tilde{f}'_v \cdot v'_y =$$
$$= \tilde{f}'_u \cdot 2y + 0 = \tilde{f}'_u \cdot 2y$$

Svar! $\underset{\sim}{f(x,y) = \dfrac{\varphi(x^2+y^2)}{e^{-x^2/2}}}$

b)

$$f(x,y) = \varphi(x^2+y^2)\, e^{x^2/2} \qquad x,y \geq 0$$

$$\boxed{f(0,y) = y^2} \qquad f(0,y) = \varphi(y^2)\, e^0 = \varphi(\underbrace{y^2}_{=t}) = \underbrace{y^2}_{=t}$$

$$\varphi(t) = t, \qquad t > 0$$

Svar: $f(x,y) = (x^2+y^2)\, e^{x^2/2}$

$$f(x,y) = \tilde{f}(u(x,y), v(x,y))$$

f

4.73 a)

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = xy$$

$$f''_{xx} = (f'_x)'_x$$
$$f''_{xy} = (f'_x)'_y$$
$$\parallel$$
$$f''_{yx} = (f'_y)'_x$$
$$f''_{yy} = (f'_y)'_y$$

$$y \neq 0 \quad \begin{cases} u = x \\ v = \dfrac{x}{y} \end{cases} \quad \left(\Leftrightarrow \begin{cases} x = u \\ y = \dfrac{x}{v} = \dfrac{u}{v} \end{cases} \right)$$

$$\boxed{f(x,y) = \tilde{f}(u(x,y), v(x,y))}$$

$$\tilde{f}'_u(u(x,y), v(x,y))$$

$$f'_x = \tilde{f}'_u \cdot \underbrace{u'_x}_{1=} + \tilde{f}'_v \cdot \underbrace{v'_x}_{1/y=} = \tilde{f}'_u + \tilde{f}'_v \cdot \frac{1}{y}$$

$$f'_y = \tilde{f}'_u \cdot \underbrace{u'_y}_{=0} + \tilde{f}'_v \cdot \underbrace{v'_y}_{= -\frac{x}{y^2}} = \tilde{f}'_v \left(-\frac{x}{y^2}\right)$$

$$f''_{xx} = (f'_x)'_x = \left(\tilde{f}'_u + \tilde{f}'_v \cdot \frac{1}{y}\right)'_x = (\tilde{f}'_u)'_x + (\tilde{f}'_v)'_x \cdot \frac{1}{y} =$$

$$= (\tilde{f}'_u)'_u \cdot \underbrace{u'_x}_{1=} + (\tilde{f}'_u)'_v \cdot \underbrace{v'_x}_{1/y=} + \left((\tilde{f}'_v)'_u \cdot \underbrace{u'_x}_{1=} + (\tilde{f}'_v)'_v \cdot \underbrace{v'_x}_{1/y=} \right) \cdot \frac{1}{y}$$

$$= \tilde{f}''_{uu} + \frac{2}{y} \tilde{f}''_{uv} + \tilde{f}''_{vv} \frac{1}{y^2}$$

$$f'_x = \tilde{f}'_u + \tilde{f}'_v \cdot \frac{1}{y}$$

$$f'_y = \tilde{f}'_v \left(-\frac{x}{y^2}\right)$$

$$\begin{cases} u = x \\ v = \dfrac{x}{y} \end{cases}$$

$$\underline{f''_{xy}} = (f'_x)'_y = \left(\tilde{f}'_u + \tilde{f}'_v \cdot \frac{1}{y}\right)'_y =$$

$$= (\tilde{f}'_u)'_y + (\tilde{f}'_v)'_y \cdot \frac{1}{y} + \tilde{f}'_v \left(-\frac{1}{y^2}\right) =$$

$$= (\tilde{f}'_u)'_u \cdot \overbrace{u'_y}^{=0} + (\tilde{f}'_u)'_v \cdot \overbrace{v'_y}^{=-\frac{x}{y^2}} + \left((\tilde{f}'_v)'_u \cdot \overbrace{u'_y}^{=0} + (\tilde{f}'_v)'_v \cdot \overbrace{v'_y}^{=-\frac{x}{y^2}}\right) \frac{1}{y}$$

$$+ \tilde{f}'_v \left(-\frac{1}{y^2}\right) =$$

$$= \underline{\tilde{f}''_{uv}\left(-\frac{x}{y^2}\right) + \tilde{f}''_{vv}\left(-\frac{x}{y^3}\right) + \tilde{f}'_v\left(-\frac{1}{y^2}\right)}$$

$$\underline{f''_{yy}} = (f'_y)'_y = \left(\tilde{f}_v\left(-\frac{x}{y^2}\right)\right)'_y = \ldots\ldots = \underline{\tilde{f}''_{vv}\frac{x^2}{y^4} + \tilde{f}'_v\left(\frac{2x}{y^3}\right)}$$

$$x^2 f''_{xx} + 2xy f''_{xy} + y^2 f''_{yy} = xy$$

$$\begin{cases} u = x \\ v = \frac{x}{y} \end{cases}$$

$$x^2\left(\tilde f''_{uu} + \cancel{\tilde f''_{uv}\frac{2}{y}} + \cancel{\tilde f''_{vv}\cdot\frac{1}{y^2}}\right)$$

$$+\ 2xy\left(\tilde f''_{uv}\cancel{\left(-\frac{x}{y^2}\right)} + \cancel{\tilde f''_{vv}\left(-\frac{x}{y^3}\right)} + \cancel{\tilde f'_v\left(-\frac{1}{y^2}\right)}\right)$$

$$+\ y^2\left(\cancel{\tilde f''_{vv}\frac{x^2}{y^4}} + \cancel{\tilde f'_v\frac{2x}{y^3}}\right) = xy$$

$$\boxed{\int g(v)\,dv = G(v) + h(u)}$$

$$\iff x^2 \tilde f''_{uu} = xy \qquad (x \neq 0)$$

$$\iff \tilde f''_{uu} = \frac{y}{x} \iff \tilde f''_{uu} = \frac{1}{v} \iff (\tilde f'_u)'_u = \frac{1}{v}$$

$$\iff \tilde f'_u = \int \frac{1}{v}\,du = \frac{u}{v} + g(v)$$

$$f = \tilde f \iff \tilde f = \int\left(\frac{1}{v}\cdot u + g(v)\right)du = \frac{1}{2v}u^2 + g(v)u + h(v)$$

$$f(x,y) = \underbrace{\frac{x^2}{2\frac{x}{y}}}_{=\frac{xy}{2}} + g\left(\frac{x}{y}\right)x + h\left(\frac{x}{y}\right)$$

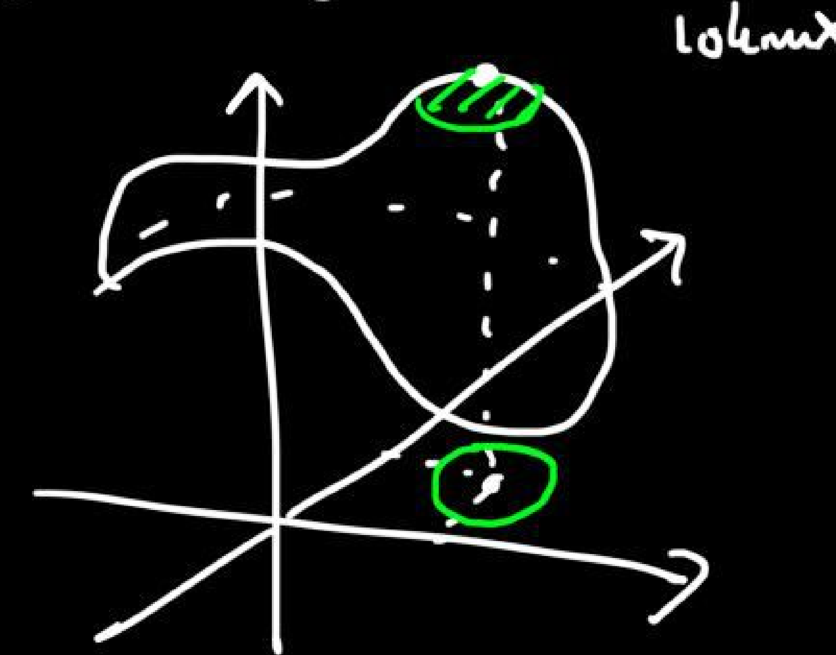
g, h 2ggr kont. deriverbara

I) $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$ stationär punkt

$\operatorname{grad} f = \bar{0}$

II) andraderivata

$z = f(x,y)$

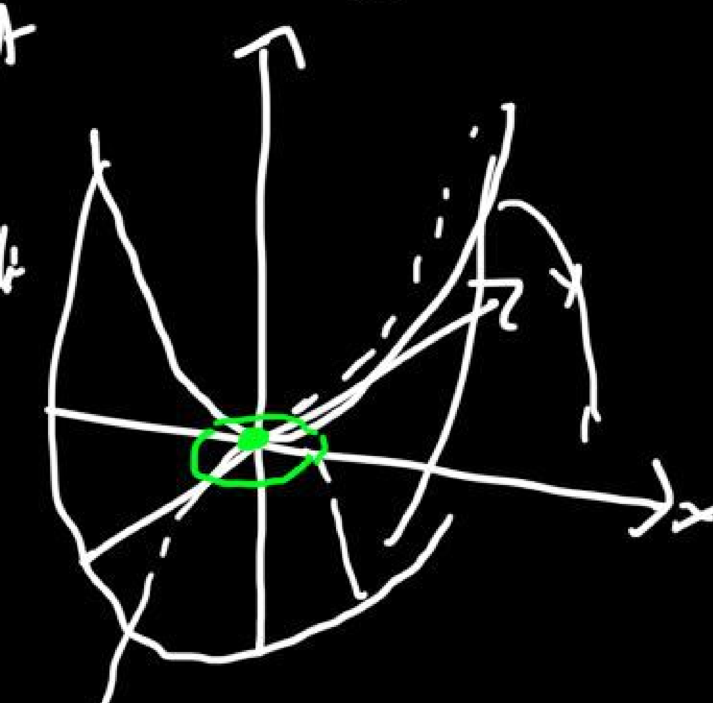


5.12. $f(x,y,z) = (x + xy + yz)e^{x}$

lok. extr. p?

$z = x^2 - y^2$

sadelpunkt
ej lok.
extr.punkt



$$\begin{cases} f'_x = (1+y)\cancel{e^{x}} + (x+xy+yz)\cancel{e^{x}} = 0 \\ f'_y = (x+z)\cancel{e^{x}} = 0 \\ f'_z = y\cancel{e^{x}} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 + x + y + xy + yz = 0 \\ x + z = 0 \\ y = 0 \end{cases}$$ kandidat!

III) $y = 0$

II) $z = -x$

I) $1 + x + 0 + 0 + 0 = 0$

$\Leftrightarrow x = -1$ ger $z = 1$, $y = 0$

Stat. punkt $(-1, 0, 1)$

$$f(x,y,z) = (x + xy + yz)e^x$$

$$\begin{cases} f'_x = (1 + x + y + xy + yz)e^x \\ f'_y = (x+z)e^x \\ f'_z = ye^x \end{cases}$$

stat. pkt

$(-1, 0, 1)$

lok. min

$+ \quad + \quad +$

$- \quad - \quad -$

lok. max

$$e^{-1}((h+k)^2 - ((k-l)^2 - l^2))$$

$$= e^{-1}(\underbrace{(h+k)^2}_{+} - \underbrace{(k-l)^2}_{-} + \underbrace{l^2}_{+})$$

sadelpunkt

$$Q(h,k,l) = f''_{xx}h^2 + f''_{yy}k^2 + f''_{zz}l^2 + 2(f''_{xy}hk + f''_{xz}hl + f''_{yz}kl)$$

$$f''_{xx} = (1+y)e^x + (1 + x + y + xy + yz)e^x$$

$$f''_{yy} = 0$$

$$f''_{zz} = 0$$

$$f''_{xy} = f''_{yx} = 1 \cdot e^x + (x+z)e^x$$

$$f''_{xz} = f''_{zx} = ye^x$$

$$f''_{yz} = e^x$$

$$Q(h,k,l) = e^{-1} \cdot h^2 + 0 + 0 + 2(e^{-1}hk + 0 + e^{-1}kl)$$

$$= e^{-1}h^2 + 2e^{-1}hk + \frac{2kl}{e^{-1}}$$

$$= e^{-1}(h^2 + 2hk + 2kl) =$$

$$= e^{-1}((h+k)^2 - k^2 + 2kl) = e^{-1}((h+k)^2 - (k^2 - 2kl))$$