Examination and Summary

Stationary stochastic processes
FMSF10/MASC04

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Examination information

The examination of FMSF10/MASC04 is Friday October 30 2020 8.00-17.00.

You are allowed to use all your material, the course book, statistical tables, and computer software for calculations (e.g. R or Matlab). You are not allowed to get help from other persons (including internet forums).

- SIGN UP for the exam as usual.
- The digital Mozquizto-exam will be open Friday October 30 8.00-17.00.
- When you have STARTED the Mozquizto-exam, the time is limited to 4 hours and you have only ONE attempt.
- The format will be similar to the preparation for the computer exercises.
- The possible grade is 3/G.
- You can prepare by calculating all exercises in the course marked by (0) and exercises 1-4 of the old exams found in Canvas.

Examination information

For possible higher grades, 4/5 and VG.

- The Mozquizto-test need to be passed!
- Additionally, you should upload solutions to the home-exam presented below within the time interval, Friday October 30 8.00-17.00.
- Finally, you will be presenting and discussing your solutions of the home-exam in a shorter oral examination meeting over Zoom.
- The possible gradings are (3/4/5) and (G/VG).
- You can prepare by calculating all exercises in the course marked by (0, +1) and exercises 4-6 of the old exams found in Canvas.

You are adviced to use the dummy assignment in Canvas, as a test to convince yourself that you can upload several pages as one pdf-file in Canvas. The test-page is open from today until the day BEFORE the exam day. This pdf will not be corrected.

Summary of the course

- Properties of a stationary stochastic process.
- Calculation of covariance functions from spectral densities.
- Characteristics of covariance function, spectral density and corresponding realization.
- Calculation of mean, covariance functions and spectral densities of filtered processes.
- Calculation of parameters, covariance functions and spectral densities of AR-process and MA-processes (pole-zero plots).
- Cross-covariance and cross-spectrum calculations.
- Calculations of derivatives and integrals.
- Optimal filters (MSE, Wiener and matched filter).
- Sampling.
- Variance calculation of the mean value estimate.
- Knowledge of basic spectral estimation techniques.

A stationary stochastic process

A continuous time process, $X(t)$, $t \in \mathbb{R}$, or a discrete time process $X_t, t = 0, \pm 1, \ldots$, is weakly stationary if,

- The expected value, $E[X(t)] = m$
- The covariance function, $C[X(s), X(t)] = r(t-s) = r(\tau)$
- The variance, $V[X(t)] = r(0)$
- The correlation function, $\rho(\tau) = r(\tau)/r(0)$

For a real-valued stationary stochastic process we have

- $V[X(t)] = r(0) \geq 0$
- $r(-\tau) = r(\tau)$
- $|r(\tau)| \leq r(0)$

Spectral density

For a weakly stationary process there exists a, positive, symmetrical and integrable spectral density function $R(f)$ such that,

$$r(\tau) = \int_{-\infty}^{\infty} R(f) e^{i2\pi f \tau} df, \quad R(f) = \int_{-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau} d\tau,$$

in continuous time, and

$$r(\tau) = \int_{-1/2}^{1/2} R(f) e^{i2\pi f \tau} df, \quad R(f) = \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i2\pi f \tau},$$

in discrete time. If $E[X(t)] = 0$ the power and the variance is

$$E[X^2(t)] = V[X(t)] = r(0) = \int_{-\infty}^{\infty} R(f) df.$$

Discrete time white noise

For discrete time white noise,

$$R(f) = \sigma^2, \quad -1/2 < f \leq 1/2,$$

and

$$r(\tau) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = \pm 1, \pm 2, \ldots \end{cases}$$

Random harmonic function

The random harmonic function

$$X(t) = A\cos(2\pi f_0 t + \phi), \quad t \in \mathbb{R},$$

is a stationary process with $A > 0$ and $\phi \in Rect(0, 2\pi)$. The covariance function is

$$r(\tau) = \frac{E[A^2]}{2}\cos(2\pi f_0 \tau), \ \ \tau \in \mathbb{R},$$

for all $f_0 > 0$ and the spectral density is defined as

$$R(f) = \frac{E[A^2]}{4}(\delta(f - f_0) + \delta(f + f_0)).$$

The random harmonic function is also defined for discrete time with $0 < f_0 \leq 0.5$

Filtering of stationary processes

For continuous time processes the output $Y(t)$, $t \in \mathbb{R}$, is obtained from the input $X(t)$, $t \in \mathbb{R}$, through

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du,$$

where $h(t)$ is the impulse response. The corresponding frequency function is

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt.$$

For discrete time processes, the integrals are changed to sums.

Filtering of stationary processes

The mean value,

$$m_Y = m_X \int_{-\infty}^{\infty} h(u)du = m_X H(0),$$

the covariance function,

$$r_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u)h(v)r_X(\tau + u - v)dudv,$$

and the spectral density,

$$R_Y(f) = |H(f)|^2 R_X(f),$$

For discrete time processes, the integrals are changed to sums.

The MA(q)-process

A moving average process of order q, MA(q), is given by

$$X_t = c_0 e_t + c_1 e_{t-1} + \ldots + c_q e_{t-q},$$

where e_t, $t = 0, \pm 1, \pm 2, \ldots$, is a zero-mean white noise Gaussian noise with variance σ^2. The expected value $m_X = 0$ and the covariance function is

$$r_X(\tau) = C[X_t, X_{t+\tau}] \neq 0 \quad |\tau| \leq q.$$

The spectral density is the discrete time Fourier transform of $r_X(\tau)$ or

$$R_X(f) = \sigma^2 |c_0 + c_1 e^{-i2\pi f} + \ldots + c_q e^{-i2\pi fq}|^2.$$

The AR(p)-process

An auto-regressive process of order p, AR(p), is given by

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t,$$

where, e_t, $t = 0, \pm 1, \pm 2 \ldots$, is zero-mean white Gaussian noise with variance σ^2. The expected value, $m_X = 0$, and the covariance function solves the Yule-Walker equations

$$r_X(\tau) + a_1 r_X(\tau - 1) + \ldots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau = 1, 2, \ldots \end{cases}$$

The spectral density is

$$R_X(f) = \frac{\sigma^2}{|1 + a_1 e^{-i2\pi f} + \ldots + a_p e^{-i2\pi f p}|^2}.$$

Example views of Mozquizto exam

Question 1 b

Consider a general AR(1)-process, $X_t + a_1 X_{t-1} = e_t$.

What is the spectral density?

- ○ $\sigma^2 a_1 \cos(2\pi f)$
- ○ $\sigma^2(1 + a_1^2 + 2a_1 \cos(2\pi f))$
- ○ $\frac{\sigma^2}{1+a_1^2+2a_1 \cos(2\pi f)}$

Example views of Mozquizto exam

Question 2

Calculate the covariance function $r_X(\tau)$, for $\tau = 0, \pm 1, \pm 2, \pm 3$, of the AR(2)-process, $X_t + X_{t-1} + 0.5X_{t-2} = e_t$, where e_t is white Gaussian noise with $E[e_t] = 0$ and $V[e_t] = 1$. Answer as a fraction or a real number with one decimal.

$r_X(0) =$ [ ] =

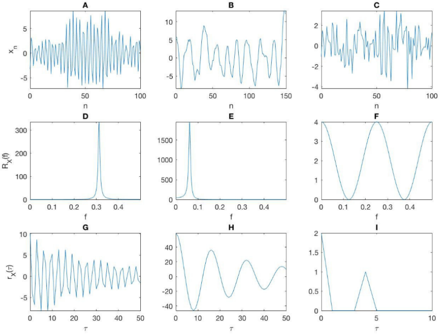
$r_X(\pm 1) =$ [ ] =

$r_X(\pm 2) =$ [ ] =

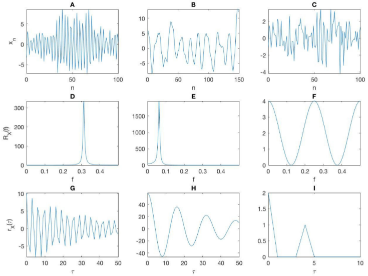
Example views of Mozquizto exam

Three different processes

The figure shows the realizations, spectral densities and covariance functions of three different discrete time models of stationary processes. Combine the figures corresponding to the same process. Specify your answers below in the order of realization, spectral density, covariance function e.g. ADG, to your choice of process 1.



Example views of Mozquizto exam



Process 1 ADG

Process 2 CFI

Process 3 BEH

Svara med ord

Cross-covariance, cross spectrum and coherence spectrum

The cross-covariance function is defined as

$$r_{X,Y}(\tau) = C[X(t), Y(t+\tau)].$$

Note that $r_{X,Y}(\tau) \neq r_{X,Y}(-\tau)$ but $r_{X,Y}(\tau) = r_{Y,X}(-\tau)$.
The complex-valued cross spectrum $R_{X,Y}(f)$ is defined so that

$$r_{X,Y}(\tau) = \int R_{X,Y}(f) e^{i2\pi f\tau} df.$$

The quadratic coherence spectrum is defined

$$\kappa^2_{X,Y}(f) = \frac{|R_{X,Y}(f)|^2}{R_X(f)R_Y(f)}, \quad 0 \leq \kappa^2_{X,Y} \leq 1.$$

The same formulations apply for discrete time processes.

Differentiation

Let $X(t)$, $t \in \mathbb{R}$, be a weakly stationary process with covariance function $r_X(\tau)$. The process is said to be differentiable in quadratic mean with the derivative $X'(t)$ if $r_X(\tau)$ is twice differentiable or if

$$V[X'(t)] = \int_{-\infty}^{\infty} (2\pi f)^2 R_X(f) df < \infty.$$

We have $m_{X'} = 0$,

$$r_{X'}(\tau) = -r_X''(\tau),$$

and

$$R_{X'}(f) = (2\pi f)^2 R_X(f).$$

The cross-covariance function, $r_{X,X'}(t, t+\tau) = r_X'(\tau)$ and $r_{X,X'}(t,t) = 0$.

Optimal filters

- Mean square error optimal filter, minimize $E[(Y(t) - S(t))^2]$.
- The matched filter for binary detection is

$$h_{opt}(u) = s(T - u),$$

  for the zero-mean white noise disturbance case. For equal decision errors, the decision level $k = s_{out}(T)/2$ with errors $\alpha = \beta = 1 - \Phi(\frac{s_{out}(T)}{2\sigma_N})$.
- The Wiener filter frequency function is

$$H_{opt}(f) = \frac{R_S(f)}{R_S(f) + R_N(f)},$$

  where $R_S(f)$ and $R_N(f)$ are the zero-mean signal and noise spectral densities respectively.

Sampling

The continuous time process $Y(t)$, $t \in \mathbb{R}$ is sampled to the discrete time sequence $Z_t = Y(t)$, $t = 0, \pm d, \pm 2d, \ldots$. The covariance function is

$$r_Z(\tau) = r_Y(\tau), \ \ \tau = 0, \pm d, \pm 2d, \ldots$$

and the spectral density

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \ \ -f_s/2 < f \leq f_s/2.$$

with $f_s = 1/d$ as the sampling frequency. With $\tau = nd$, $r_Z(\tau)$ is converted to $r_X(n)$, $n = 0, \pm 1, \pm 2, \ldots$, and

$$R_X(\nu) = f_s \cdot R_Z(\nu f_s),$$

for $\nu = f \cdot d = f/f_s$.

Estimation of mean

If $X(t)$, $t = 1, 2 \ldots$ is weakly stationary with the unknown expected value m then

$$\hat{m}_n = \frac{1}{n} \sum_{t=1}^{n} X(t)$$

is an unbiased estimate of m as $E[\hat{m}_n] = m$. The variance is

$$V[\hat{m}_n] = C[\frac{1}{n} \sum_{t=1}^{n} x(t), \frac{1}{n} \sum_{s=1}^{n} x(s)] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r(\tau).$$

For large n,

$$V[\hat{m}_n] \approx \frac{1}{n} \sum_{\tau} r(\tau).$$

Spectrum estimation

The periodogram is defined as

$$\widehat{R}_x(f) = \frac{1}{n}|\sum_t x(t)w(t)e^{-i2\pi ft}|^2,$$

where $w(t)$ is a data window. With the Hanning window, the spectrum estimate will have better leakage properties (lower sidelobes), although the resolution is somewhat degraded, (wider mainlobe), in comparison to the rectangular window.

The variance of the periodogram is

$$V[\widehat{R}_x(f)] \approx R_X^2(f) \;\; 0 < |f| < 0.5$$

Dividing the sequence into K possibly overlapping sequences and calculating the average of K approximately uncorrelated estimates (Welch method), reduces the variance to

$$V[\widehat{R}_{mv}(f)] \approx \frac{1}{K} R_X^2(f).$$