Lecture 6: Linear Quadratic Control

- Optimal State Feedback $u_k = -Lx_k$
- Optimal State Estimation $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$
- Optimal Output Feedback $u_k = -L\hat{x}_{k|k-1}$ and $u_k = -L\hat{x}_{k|k}$

Chapters 6,7,8

Math Repetition

Suppose the matrix Q is symmetric: $Q = Q^T$. Then

- $Q > 0$ means that $x^T Q x > 0$ for any $x \neq 0$
  - True iff all eigenvalues of Q are positive.
  - We say that Q is positive definite.
- $Q \geq 0$ means that $x^T Q x \geq 0$ for any x
  - True iff all eigenvalues of Q are non-negative.
  - We say that Q is positive semidefinite.

Math Repetition

The trace of a matrix is the sum of all diagonal elements:

$$\text{trace } Q = \sum_{i}^{n} Q_{ii}$$

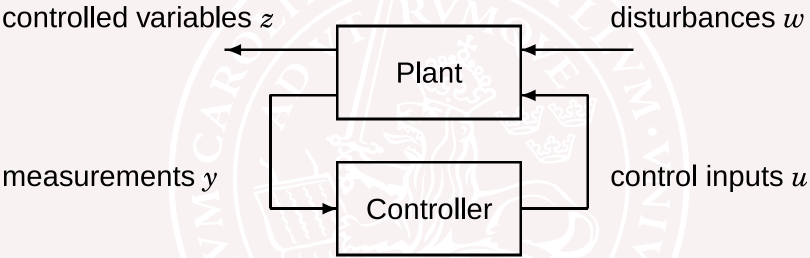
A useful property of the matrix trace:

$$\text{trace } ABC = \text{ trace } CAB = \text{ trace } BCA$$

Parseval's formula: Suppose that f_k and g_k have finite energy and that their Z-transforms are $F(z)$ and $G(z)$, respectively. Then

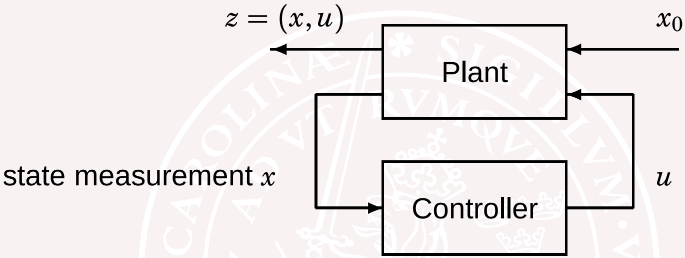
$$\sum_{0}^{\infty} f_k^* g_k = \int_0^{2\pi} F(e^{i\omega})^* G(e^{i\omega}) d\omega / 2\pi$$

A General Optimization Setup



The objective is to find a controller that optimizes the transfer matrix $G_{zw}(z)$ from disturbances w to controlled outputs z.

First problem: State Feedback



Minimize $$\sum_{0}^{\infty} \left(x_k^T Q_1 x_k + 2 x_k^T Q_{12} u_k + u_k^T Q_2 u_k \right)$$

subject to $$x_{k+1} = A x_k + B u_k, \qquad x_0 \text{ given}$$

Note: We assume $u_c = 0$ in this lecture

Dynamic programming, Richard E. Bellman 1957

An optimal trajectory on the time interval $[k, N]$ must be optimal also on each of the subintervals $[k, k+1]$ and $[k+1, N]$.

Dynamic programming in linear quadratic control

An optimal trajectory on the time interval $[k, k+1]$ must be optimal also on each of the subintervals $[k, k+1]$ and $[k+1, N]$.

Let $x^T S_k x$ be the optimal cost on the time interval $[k, \infty]$:

$$x^T S_k x = \min_u \sum_k^N \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \qquad \text{with } x_k = x$$

Dynamic programming in linear quadratic control

$$x_k = x, \qquad x_{k+1} = Ax_k + Bu_k$$

$$\begin{aligned}
x^T S_k x &= \min_u \sum_k^N \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\
&= \min_u \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \sum_{k+1}^N \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\} \\
&= \min_u \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \Big(Ax + Bu\Big)^T S_{k+1} \Big(Ax + Bu\Big) \right\}
\end{aligned}$$

by definition of S. This gives **Bellman's equation**:

$$x^T S_k x = \min_u \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + (Ax + Bu)^T S_{k+1} (\dots)$$

Completion of squares

The scalar case: Suppose $c > 0$.

$$ax^2 + 2bxu + cu^2 = x\left(a - \frac{b^2}{c}\right)x + \left(u + \frac{b}{c}x\right)c\left(u + \frac{b}{c}x\right)$$

is minimized by $u = -\frac{b}{c}x$. The minimum is $\left(a - b^2/c\right)x^2$.

The matrix case: Suppose $Q_u > 0$. Then

$$\begin{aligned}&x^T Q_x x + 2x^T Q_{xu} u + u^T Q_u u\\ &= x^T(Q_x - Q_{xu}Q_u^{-1}Q_{xu}^T)x + (u + Q_u^{-1}Q_{xu}^T x)^T Q_u (u + Q_u^{-1}Q_{xu}^T x)\end{aligned}$$

is minimized by $u = -Q_u^{-1}Q_{xu}^T x$.
The minimum is $x^T(Q_x - Q_{xu}Q_u^{-1}Q_{xu}^T)x$.

The Riccati Equation

Completion of squares in Bellman's equation gives

$$x^T S_k x = \min_u x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u + \left(Ax + Bu\right)^T S_{k+1}\left(\ \dots\ \right)$$
$$= x^T \left(Q_1 + A^T S_{k+1} A - (A^T S_{k+1} B + Q_{12})(Q_2 + B^T S_{k+1} B)^{-1} (\dots)^T \right)$$

with minimum attained for

$$u = -(Q_2 + B^T S_{k+1} B)^{-1} (A^T S_{k+1} B + Q_{12})^T x$$

The equation

$$S_k = Q_1 + A^T S_{k+1} A - (A^T S_{k+1} B + Q_{12})(Q_2^{-1} + B^T S_{k+1} B)(\dots)^T$$

is called the discrete time *Riccati equation*

Jocopo Francesco Riccati, 1676–1754

Linear Quadratic Optimal Control

Problem ($N = \infty$):

Minimize $$\sum_{0}^{\infty} \left(x_k^T Q_1 x_k + 2x_k^T Q_{12} u_k + u_k^T Q_2 u_k \right)$$

subject to $$x_{k+1} = Ax_k + Bu_k, \qquad x_0 \text{ given}$$

Solution: Assume (A, B) controllable. Then there is a unique $S > 0$ solving the Riccati equation

$$S = Q_1 + A^T SA - (A^T SB + Q_{12})(Q_2 + B^T SB)^{-1}(A^T SB + Q_{12})^T$$

The optimal control law is $u = -Lx$ with

$$L = (Q_2 + B^T SB)^{-1}(A^T SB + Q_{12})^T$$

The minimal value is $x_0^T S x_0$.

Remark: The feedback gain L does not depend on x_0

Example: First order system

For $x_{k+1} = x_k + u_k$, x_0 given,

Minimize
$$\sum_{0}^{\infty} \left(x_k^2 + \rho u_k^2 \right)$$

Riccati equation
$$S = 1 + S - \frac{S^2}{\rho + S} \quad \Rightarrow \quad S = \frac{1}{2} + \sqrt{\frac{1}{4} + \rho}$$

Controller
$$L = \frac{1}{\frac{1}{2} + \sqrt{\frac{1}{4} + \rho}}, \quad u = -Lx$$

Closed loop
$$x_{k+1} = (1 - L)x_k$$

Optimal cost
$$\sum_{0}^{\infty} \left(x^2 + \rho u^2 \right) dt = x_0^T S x_0$$

What values of ρ give the fastest response? Why?

Theorem: Stability of the closed-loop system

Assume that

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$$

is positive definite and that there exists a positive-definite steady-state solution S to the algebraic Riccati equation. Then the optimal controller $u_k = -Lx_k$ gives an asymptotically stable closed-loop system $x_{k+1} = (A - BL)x_k$.

Proof sketch: If $x_k \neq 0$ then

$$x_{k+1}^T S x_{k+1} - x_k^T S x_k = -\begin{pmatrix} x_k \\ u_k \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} < 0$$

Hence $x_k^T S x_k$ is decreasing and tends to zero as $k \to \infty$.

LQ design in Matlab

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DLQR Linear-quadratic regulator design for discrete-time systems

[K,S,E] = DLQR(A,B,Q,R,N)  calculates the optimal gain matrix K
 such that the state-feedback law  u[n] = -Kx[n]  minimizes the
 cost function

      J = Sum {x'Qx + u'Ru + 2*x'Nu}

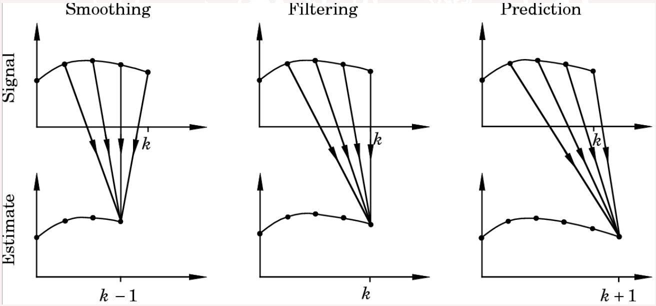
 subject to the state dynamics   x[n+1] = Ax[n] + Bu[n].

The matrix N is set to zero when omitted. Also returned are the
Riccati equation solution S and the closed-loop eigenvalues E:
                                 -1
A'SA - S - (A'SB+N)(R+B'SB) (B'SA+N') + Q = 0,   E = EIG(A-B*K).
```

Prediction and filtering

* Wiener (1949) Stationary I/O case
* Kalman and Bucy (1960) Time-varying state-space

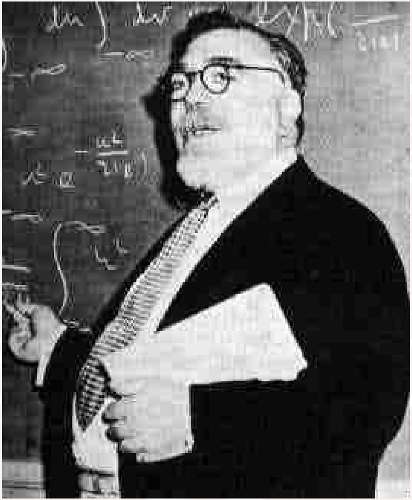
Estimate $x(k+m)$ given $\{y(i), u(i)\,|\,i \le k\}$



Rudolf Kalman, (born 1930)

Recipient of the 2008 Charles Stark Draper Prize from the US National Academy of Engineering "for the devlopment and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products.”

Norbert Wiener, 1894–1964



Problem Formulation

Given the system

$$\begin{aligned} x_{k+1} &= Ax_k + Bv_k, \\ y_k &= Cx_k + e_k \end{aligned}$$

where v and e are white noise with

$$E \begin{bmatrix} v_k \\ e_k \end{bmatrix} \begin{bmatrix} v_k \\ e_k \end{bmatrix}^T = \begin{bmatrix} Q_v & Q_{ve} \\ Q_{ve}^T & Q_e \end{bmatrix}$$

Want to minimize $E|M(x - \hat{x})|^2$. Solution (indepent of M):

$$\begin{aligned} \widehat{x}_{k|k-1} &= E(x| \text{ measurements up to time k-1}) \\ \widehat{x}_{k|k} &= E(x| \text{ measurements up to time k}) \end{aligned}$$

Optimal Filtering and Prediction

The filtered state estimate $\widehat{x}_{k|k}$ and predicted state $\widehat{x}_{k+1|k}$ can be obtained in a two step procedure

Error update:

$$\widehat{x}_{k|k} = \widehat{x}_{k|k-1} + \kappa(y_k - C\widehat{x}_{k|k-1})$$

Time update:

$$\widehat{x}_{k+1|k} = A\widehat{x}_{k|k} + Bu_k$$

where the stationary solution is (if $Q_{ve} = 0$)

$$\kappa = PC^T(Q_e + CPC^T)^{-1}$$

where $P = \mathrm{cov}(x - \hat{x}_{k|k-1})$ is found from the Riccati equation

$$P = APA^T + Q_v - APC^T(Q_e + CPC^T)^{-1}CPA^T$$

Matlab - Optimal Kalman Filters

```
DLQE  Kalman estimator design for discrete-time systems.

    Given the system

       x[n+1] = Ax[n] + Bu[n] + Gw[n]     {State equation}
       y[n]   = Cx[n] + Du[n] +  v[n]     {Measurements}

    with unbiased process noise w[n] and measurement noise v[n]
    with covariances

       E{ww'} = Q,    E{vv'} = R,    E{wv'} = 0 ,

    [M,P,Z,E] = DLQE(A,G,C,Q,R)  returns the gain matrix M such
    that the discrete, stationary Kalman filter with observation
    and time update equations

       x[n|n]   = x[n|n-1] + M(y[n] - Cx[n|n-1] - Du[n])
       x[n+1|n] = Ax[n|n] + Bu[n]

    produces an optimal state estimate x[n|n] of x[n] given y[n] and
    the past measurements.  The resulting Kalman estimator can be
    formed with DESTIM.

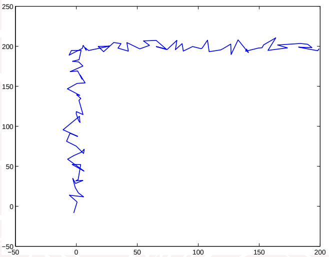
    Also returned are the steady-state error covariances

       P = E{(x[n|n-1] - x)(x[n|n-1] - x)'}     (Riccati solution)
       Z = E{(x[n|n] - x)(x[n|n] - x)'}         (Error covariance)

    and the estimator poles E = EIG(A-A*M*C).
```

Example – GPS tracking

Position measurements + noise with 5 meter standard deviation



Clearly this is not the true path, can we improve the estimate?

Assume this motion model for coordinates x, y

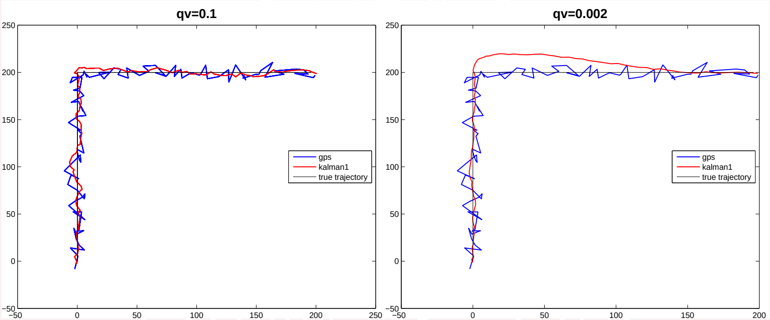
$$
\begin{aligned}
x(k+1) &= x(k) + h v_x(k) \\
v_x(k+1) &= v_x(k) + \text{noise} \\
y(k+1) &= y(k) + h v_y(k) \\
v_y(k+1) &= v_y(k) + \text{noise}
\end{aligned}
$$

Example – GPS tracking

```
ygps = xtrue + gps_accuracy*randn(size(xtrue));

% The Kalman filter (matlab notation)
I = eye(2);
A = [I h*I; 0*I I];
G = [0*I;I];
C = [I 0*I];
Q = qv*I;             % tuning parameter
R = gps_accuracy^2*I;
[M,P,Z,E] = dlqe(A,G,C,Q,R);
K=A*M;
kalman1 = ss(A-K*C,K,C,0*I,h);
% Some plotting
time = (0:2*N)*h;
xpred = lsim(kalman1, ygps,time)';
```

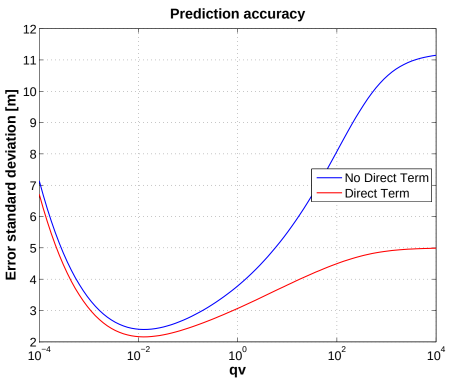
Example – Results



Trade off between rapid tracking and good noise reduction

High Q_v/Q_e: Rapid tracking
Low Q_v/Q_e: Good noise reduction

Prediction Accuracy



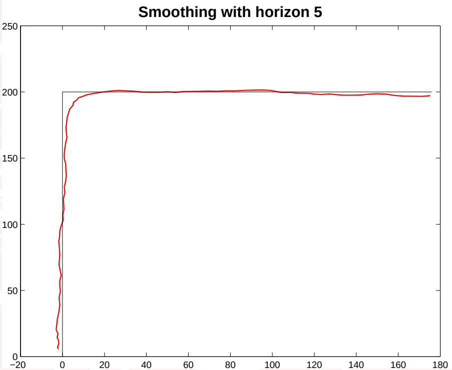
Best q_v tradeoff between tracking speed vs noise reduction

Direct term in filter gives some performance improvement.

Smoothing would give even better results.

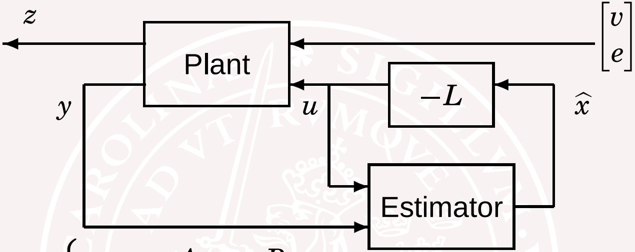
Smoothing

Performance with fixed lag smoothing (horizon 5) $\hat{x}_{k|k+5}$



Improvement of about 10% in standard deviation

Output feedback



Plant: $\begin{cases} x_{k+1} = Ax_k + Bu_k + v_k \\ y_k = Cx_k + e_k \end{cases}$

Controller: $\begin{cases} \widehat{x}_{k+1|k} = A\widehat{x}_{k|k-1} + Bu_k + K[y_k - C\widehat{x}_{k|k-1}] \quad (K = A\kappa) \\ u_k = -L\widehat{x}_{k|k-1} \text{ no direct term, or} \\ u_k = -L\widehat{x}_{k|k} \text{ direct term} \end{cases}$

Minimizes $\mathbf{E}|z|^2 = \mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$

when v, e are white uncorrelated noise

The idea of separation

The state feedback control law is independent of the noise covariance

The Kalman filter minimizes $\mathbf{E}|M\tilde{x}|^2$ independently of M

This makes it possible to optimize the control law $u(t) = -L\hat{x}(t)$ and the estimator separately.

Duality between control and estimation

Optimal control	State estimation
A	A^T
B	C^T
Q_1	Q_v
Q_2	Q_e
Q_{12}	Q_{ve}
S	P
L	K^T

The Resulting Controller (if $Q_{ve} = 0$)

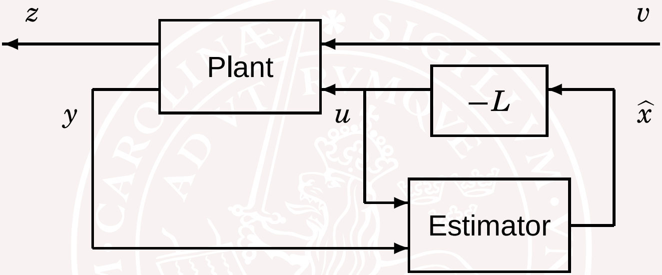
$$
\begin{aligned}
U &= -L(qI - A_1)^{-1}KY \quad \text{no direct term} \\
U &= -Lq(qI - A_0)^{-1}\kappa Y \quad \text{direct term}
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= A - BL - KC \quad \text{no direct term} \\
A_0 &= (I - \kappa C)(A - BL) \quad \text{direct term}
\end{aligned}
$$

where $\kappa = PC^T(Q_e + CPC^T)^{-1}$ and $K = A\kappa$

Output feedback - Analysis



Plant: $$\begin{cases} x_{k+1} = Ax_k + Bu_k + v_k \\ y_k = Cx_k + e_k \end{cases}$$

Controller: $$\begin{cases} \widehat{x}_{k+1|k} = A\widehat{x}_{k|k-1} + Bu_k + K[y_k - C\widehat{x}_{k|k-1}] \\ u_k = -L\widehat{x}_{k|k-1} \end{cases}$$

Closed loop dynamics

Eliminate u and y:

$$x_{k+1} = Ax_k - BL\widehat{x}_k + v_k$$
$$\widehat{x}_{k+1|k} = A\widehat{x}_{k|k-1} - BL\widehat{x}_{k|k-1} + K[Cx_k - C\widehat{x}_{k|k-1}] + Ke_k$$

Introduce $\widetilde{x} = x - \widehat{x}$

$$\begin{bmatrix} x_{k+1} \\ \widetilde{x}_{k+1|k} \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x_k \\ \widetilde{x}_{k|k-1} \end{bmatrix} + \begin{bmatrix} v_k \\ v_k - Ke_k \end{bmatrix}$$

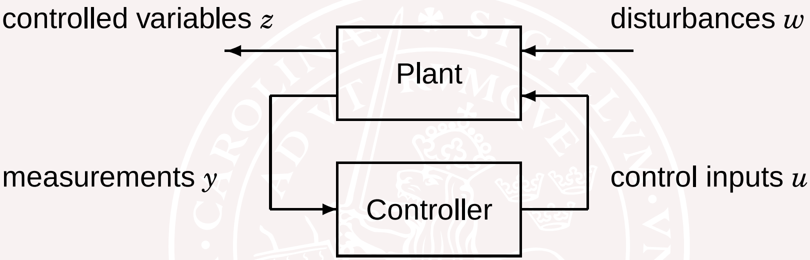
Two kinds of closed loop poles

Process poles: $0 = \det(zI - A + BL)$

Observer poles: $0 = \det(zI - A + KC)$

May also use observer with direct term, $\widehat{x}_{k|k}$

Summary: Linear Quadratic Gaussian Control (LQG)



For a linear plant, minimize a quadratic function of the map from disturbance w to controlled variable z

$$\text{Minimize} \quad \text{trace} \int_0^{2\pi} Q G_{zw}(e^{i\omega}) G_{zw}(e^{i\omega})^* d\omega$$

Two interpretations of this criterion...

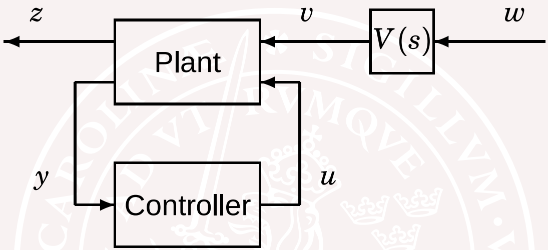
Impulse response interpretation of LQG

Let $g_{zw}(k)$ be the impulse response corresponding to the transfer function $G_{zw}(z)$. Then

$$\text{trace}\,\frac{1}{2\pi}\int_0^{2\pi} QG_{zw}(e^{i\omega})G_{zw}(e^{i\omega})^* d\omega = \text{ trace}\sum_0^{\infty} Qg_{zw}(k)g_{zw}(k)^*$$

so LQG control minimizes the impulse response “energy”.

Stochastic interpretation of LQG



$$\|z\|_Q^2 = \mathbf{E}(z^T Q z) = \text{trace } \frac{1}{2\pi}\int_{-\infty}^{\infty} Q G_{zw}(e^{i\omega}) G_{zw}(e^{i\omega})^* d\omega$$

is the weighted output variance when the input v has spectral density $\Phi_v(\omega) = V(i\omega)V(i\omega)^*$. Hence the output variance can be minimized by defining $G_{zw}(i\omega) = G_{zv}(i\omega)V(i\omega)$ and solving the LQG problem

$$\text{Minimize} \quad \text{trace} \int_{-\infty}^{\infty} Q G_{zw}(e^{i\omega}) G_{zw}(e^{i\omega})^* d\omega$$