Assignment 5

Handin solutions using the Canvas system

1. A particle influenced by the gradient flow from a potential $V(x)$, where $x \in R^n$, and thermal noise whose strength is $D = k_B T$, is described by the SDE

$$dX_t = -\nabla V(X_t)dt + \sqrt{2D}\, dW_t.$$

Here W_t is a Wiener process with incremental variance $E(dW_t)^2 = dt$. The corresponding Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = \nabla \cdot ((\nabla V)p) + D\Delta p$$

a. Find the stationary solution $p(x)$ when $V = \alpha x^2$ (scalar case with $x \in R$)

b. Show (for the general case) that a stationary solution to the FP-equation is given by the **Gibbs distribution** (assuming $V(x)$ is sufficiently nice)

$$p(x) = \frac{1}{Z} e^{-V(x)/D},$$

where Z is a normalization factor, called the **partition function**.

2. Consider the Black-Scholes equations for option pricing

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

with the known boundary condition $V(S,T) = g(S)$. Integrating the PDE backwards in time gives the value of the option as a function of the current stock price S and time t with risk-free interest rate r and volatility σ.

a. Show that the transformation $\tau = T - t$, $x = \log(S)$ and

$$u(x,\tau) = e^{\alpha x + \beta\tau} V$$

with suitable α, β will transform the BS equation to the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$$

and determine the boundary conditions on u when $g(S) = \max(S-1, 0)$.

b. Solve the BS equations numerically for the case $g(S) = \max(S-1,0)$, $T = 1$, $\sigma = 0.4$, $r = 0.04$. Either find a suitable SW package or implement a simple PDE solver, e.g using a simple approximation scheme yourself, such as

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{\sigma^2}{2}\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} \quad \text{(Explicit Euler)}$$

where $u_i^j \approx u(x_i, \tau_j)$. Handin code and plots of $V(S,t)$ for suitable S and t.

Hint: `http://www.iam.fmph.uniba.sk/institute/stehlikova/fd14en/ex/ex10.html`