Assignment 2

Handin solutions to these problems within two weeks, using the Canvas system

1. Stochastic Integral Consider the stochastic Ito integral

$$I(t) = \int_0^t w(s)dw(s)$$

where w is a Wiener process with $w(0) = 0$ and unit incremental variance, i.e. $E(dw^2) = dt$. The integral is by definition constructed as a limit as $N \to \infty$ of

$$I_N(t) = \sum_{k=1}^{N} w(t_k)(w(t_{k+1}) - w(t_k))$$

where $t_0 = 0, t_{N+1} = t$ and $t_{k+1} - t_k = t/N$.

a. Show that with $I(t) := \lim_{N\to\infty} I_N(t)$ we have (with probability 1)

$$I(t) = \frac{w(t)^2 - t}{2}$$

b. Verify the result in a) numerically, using the code to generate a Wiener process w that you wrote in Assignment 1. Calculate $I_N(t)$ for a large N and compare with the histograms of $I(t)$ resulting from the formula in a). Verify that you get roughly the same result.

2. A double integrator system The SDE

$$dx = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw,$$

where $x(t) = \begin{bmatrix} p(t) & v(t) \end{bmatrix}^T$ describes position and velocity variations of a particle influenced by random acceleration noise.

a. Show that the SDE has the solution

$$x(h) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(0) + d(h)$$

where $d(h)$ is a zero mean Gaussian random vector and calculate the covariance matrix $P(h) = E(d(h)d(h)^T)$.

b. Calculate the correlation coefficient between position and velocity fluctuations

$$\rho_{pv} = \frac{\text{Cov(p(h),v(h))}}{\sqrt{\text{Var(p(h))Var(v(h))}}}?$$

Does it depend on h?

3. Let $x(t)$ be Brownian motion in R^n, i.e. $dx = dw$ and consider the distance function $r(x) = |x|$. Show that $r(t)$ satisfies the scalar SDE

$$dr = \frac{n-1}{2r} dt + e_r(w(t)) \cdot dw$$

where $e_r(x) := x/|x|$ is a unit vector in R^n. (Hint: Use Ito's lemma, and $\nabla r(x) = e_r(x)$).