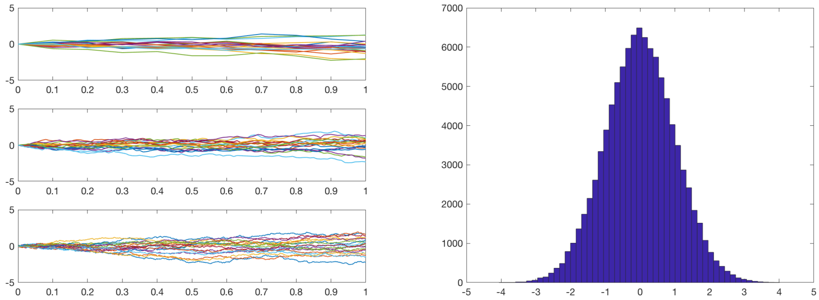
Assignment 1

Handin solutions to these problems within two weeks, using the Canvas system

1. **Simulating Brownian Motion** Consider standard Brownian motion $w(t)$, where $w(0) = 0$ with independent increments $dw = w(t+dt) - w(t) \in N(0, dt)$. Write code to generate $N = 10^5$ trajectories sampled at a rate $h = 0.1, 0.01$ and 0.001 for $t = [0, h, 2h, \ldots, T]$.

From theory we should have $x(T) \in N(0, T)$. Verify that your simulations give this distribution, by e.g. plotting a histogram and calculating the mean and variance of $x(T)$. You can use $T = 1$. Verify that the result does not depend on h. Interpret the results.

Extra: Also plot the running maximum $M(t) := \max_{s \in [0,t]} w(s)$. Can you guess the distribution of $M(t)$? Challenge: Prove your guess.



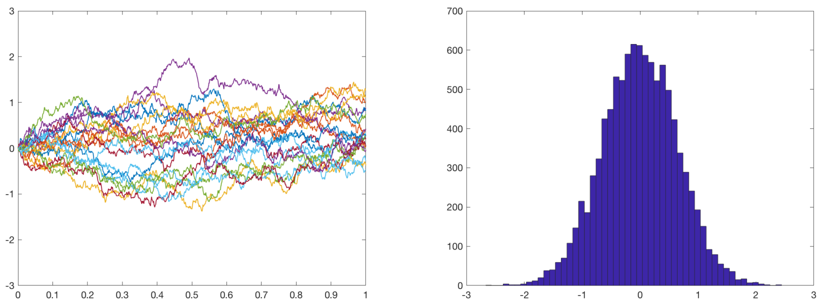
2. **Simulating a stochastic differential equation** Generate discrete-time simulations of the continuous time stochastic differential equation

$$dx = -x(t)dt + dw, \qquad x(0) = 0,$$

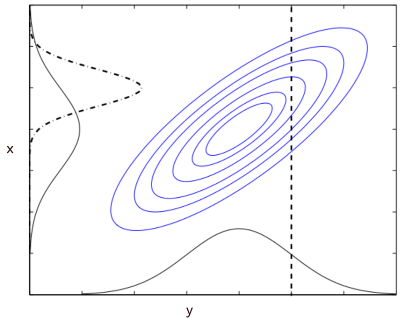
where as before $dw \in N(0, dt)$ ("a stable first order system driven by white noise with unit incremental variance"). Generate a large number, e.g. $N = 10^5$, of discrete-time approximate trajectories $x_h(kh)$, using

$$x_h(kh + h) = x_h(kh) + dx$$

using a small value for $h = dt$, e.g. $h = 0.001$, and final time $T = 1$. Verify numerically that $x(1) \in N(0, p)$. Can you guess p (the variance of $x(1)$ obtained in the limit when $h \to 0$) ?



3. Let x and y be Gaussian distributed with $\begin{bmatrix} x \\ y \end{bmatrix} \in N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_{yy} \end{bmatrix}\right)$



a. Prove that the distribution of x conditioned on a measurement y is given by

$$x \mid y \in N(m, \Sigma), \quad \text{where} \quad \begin{cases} m = m_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - m_y) \\ \Sigma = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}^T \end{cases}$$

b. Express the same result using the information matrix instead, where

$$\begin{bmatrix} I_{xx} & I_{xy} \\ I_{xy}^T & I_{yy} \end{bmatrix} := \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_{yy} \end{bmatrix}^{-1}$$

c. Let $w(t)$ be Brownian motion with unit incremental variance. Assume we know $w(t_1)$ and $w(t_2)$. Describe the conditional distribution of $w(t)$ given $w(t_1)$ and $w(t_2)$, where $t_1 < t < t_2$. (The result can be used to generate fast code for zooming in on part of a Wiener process.)

4. Assume $x(t)$ is standard Brownian motion, with $x(0) = 0$ and unit incremental variance, and consider $x_i := x(t_i)$ with $0 < t_1 < t_2 \ldots < t_n$.

a. Prove that $E(x_i x_j) = \min(t_i, t_j)$ and conclude that the covariance function is

$$C = \begin{bmatrix} t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & \ldots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & \ldots & t_n \end{bmatrix}.$$

The distribution function of x is hence given by

$$p(x_1, x_2, \ldots, x_n) = \frac{1}{(\det(2\pi C))^{1/2}} \exp\left(-\frac{1}{2}x^T C^{-1} x\right).$$

Show also that

$$\det(C) = t_1(t_2 - t_1)\ldots(t_n - t_{n-1}).$$

b. Find a formula for the information matrix $I := C^{-1}$. Hint: It turns out to be symmetric and have tri-diagonal structure (Can you understand why?). Another hint: You might want to prove and use the formula

$$I_{ij} = -E\left[\frac{\partial^2}{\partial x_i \partial x_j} \log p(x)\right].$$