Three-body problem : Lagrangian points

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Problem

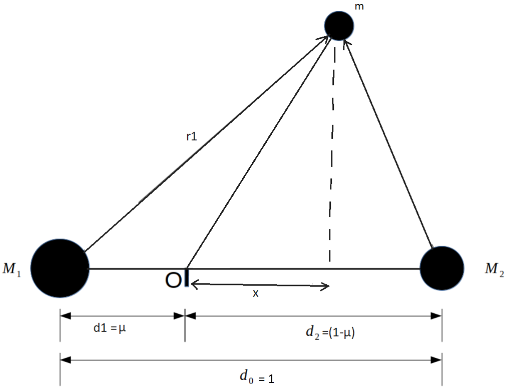


Figure 1: Set up three body problem

Equations of motion

- We normalize the distances and masses to have a dimensionless set of equations:

$$\begin{cases} \mu = \dfrac{M_2}{M_1 + M_2} \text{ (reduced mass)} \\ -M_2 d_2 + M_1 d_1 = 0 \text{ (center of mass formula)} \\ M_1 + M_2 = M = 1 \\ M_1 = 1 - \mu \end{cases}$$

$$\begin{cases} -(1-\mu)d_2 + \mu d_1 = 0 \\ d_1 + d_2 = 1 \end{cases} \Leftrightarrow \begin{cases} d_1 = \mu \\ d_2 = 1 - \mu \end{cases} (1)$$

Equations of motion

We use (1) into the rotational velocity to normalize the time :

$$\omega^2 = \frac{G(M_1 + M_2)}{d_0^3} = 1$$

- Calculate kinetic energy E_k and potential energy E_p in the co-rotating frame R'
- Calculate the Lagrangian $L = E_k - E_p$.
- Solve the Euler-Lagrange Equations of motion and we obtain:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} \tag{3}$$

$$\begin{cases} \ddot{x} = -(1-\mu)\dfrac{(x+\mu)}{((x+\mu)^2+y^2)^{\frac{3}{2}}} - \mu\dfrac{(x-1+\mu)}{((x-1+\mu)^2+y^2)^{\frac{3}{2}}} + x + 2\dot{y} \\ \ddot{y} = -(1-\mu)\dfrac{y}{((x+\mu)^2+y^2)^{\frac{3}{2}}} - \mu\dfrac{y}{((x-1+\mu)^2+y^2)^{\frac{3}{2}}} + y - 2\dot{x} \end{cases}$$

Equations of motion

For some fixed point $\ddot{x}=\ddot{y}=0 \Leftrightarrow$

$$\begin{cases} 0 = \dfrac{-(1-\mu)(x+\mu)}{r_1^3} - \dfrac{\mu(x-1+\mu)}{r_2^3} + x \\ 0 = \dfrac{-(1-\mu)y}{r_1^3} - \dfrac{\mu y}{r_2^3} + y \end{cases}$$

$\longrightarrow$ <u>Trivial case</u> y=0: we find 3 real solutions $\longrightarrow$ the system gives:

$$\begin{aligned} 0 &= \frac{-\mu(x-1+\mu)}{|x-1+\mu|^3} - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} + x \\ \Leftrightarrow 0 &= x(x-1+\mu)^2(x+\mu)^2 - \mu(x+\mu)^2 sign(x-1+\mu) \\ &\quad - (1-\mu)sign(x+\mu)(x-1+\mu)^2 \end{aligned} \tag{4}$$

Find Lagrangian points and check their stability

Case $y = 0$ with (x,y) L_1, $L2$ or $L3$ with different solution depending on the sign:

x	$-\infty$		$-\mu$		0		$1-\mu$		$+\infty$
$sgn(x-1+\mu$		$-$	0	$-$	0	$-$	0	$+$	
$sgn(x+\mu)$		$-$	0	$+$	0	$+$	0	$+$	
combinations		- and -	0	- and +	0	- and +	0	+ and +	

$$
\begin{aligned}
0 = {} & x(x-1+\mu)^2(x+\mu)^2 - \mu(x+\mu)^2 sign(x-1+\mu) \\
& -(1-\mu)sign(x+\mu)(x-1+\mu)^2
\end{aligned}
$$

Lagrangian points for L1, L2 and L3

We know that L1, L2, L3 are aligned.

- L2 : On the line defined by the mass, beyond the heavier one.

Here we considered $sign(x-1+\mu)=+$; $x=1-\mu+\epsilon$ $sign(x+\mu)=+$
$sign(x+\mu)=+$:

$$0 = x(x+\mu)^2(x-1+\mu)^2 - \mu(x+\mu)^2 - (1-\mu)((x-1+\mu)^2 \tag{5}$$

$$0 = (1-\mu+\epsilon)(\epsilon+1)^2\epsilon^2 - \mu(\epsilon+1)^2 - (1-\mu)\epsilon^2 \tag{6}$$

$$\epsilon = +(\frac{\mu}{3})^{1/3} \tag{7}$$

$$L2 = (1-\mu+\left(\frac{\mu}{3}\right)^{1/3}, 0) \tag{8}$$

Lagrangian points for L1, L2 and L3

- L1 : On the line defined by the mass, between the two mass M1 and M2.

Here we considered $sign(x - 1 + \mu) = -$; and again $x = 1 - \mu + \epsilon$:

$$\epsilon = -(\frac{\mu}{3})^{1/3} \tag{9}$$

$$L1 = (1 - \mu - \left(\frac{\mu}{3}\right)^{1/3}, 0) \tag{10}$$

Lagrangian points for L1, L2 and L3

- L3: On the line defined by the mass, beyond the lightest one.

Now, we have to assume that $sign(x - 1 + \mu) = -$, $sign(x + \mu) = -$ and $x = -1 + \mu + \epsilon$

$$\epsilon = -\frac{17\mu}{12} \tag{11}$$

$$L3 = (-1 - \frac{5\mu}{12}, 0) \tag{12}$$

Stability of L1, L2 and L3

To find the stability, let assume: $\begin{cases} \text{x=x}_i + \epsilon q_1 \\ \text{y=y}_i + \epsilon q_2 \end{cases}$

By inserting in the Euler-Lagrange function x and y (2) and (3) we get :

$$\begin{cases} \ddot{q_1} = 2\dot{q_2} - 3q_1 \\ \ddot{q_2} = -2\dot{q_1} - \dfrac{3}{2}q_2 \end{cases} \tag{13}$$

Finally, we get:

$$\begin{cases} q_n = A_i e^{i\Omega t}, & n = 1, 2 \\ with & \Omega^2 = -\dfrac{1}{4}[1 \pm i\sqrt{71}] \end{cases} \tag{14}$$

Lagrangian points L4 and L5

Case $y \neq 0$ with (x,y) L_4 or L_5: Vector projection with $\vec{u_\perp} = (y, -x)$ and $\vec{u_{||}} = (x, y)$

$$\begin{aligned} \vec{f(x, y)}.\vec{u_\perp} &= 0 \\ \vec{f(x, y)}.\vec{u_{||}} &= 0 \end{aligned} \tag{15}$$

$$\iff \begin{cases} 0 = (1-\mu)\dfrac{(x+\mu)y - xy}{r_1^3} + \mu\dfrac{(x-1+\mu)y - xy}{r_2^3} \\ 0 = -x^2 - y^2 + (1-\mu)\dfrac{(x+\mu)x + y^2}{r_1^3} + \mu\dfrac{(x-1+\mu)x - y^2}{r_2^3} \end{cases} \tag{16}$$

Lagrangian points L4 and L5

At the end we obtained :

$$\Longleftrightarrow \begin{cases} r_1 = r_2 \\ r_1 = r_2 = 1 \end{cases} \tag{17}$$

- L4 and L5 are the edges of two equilateral triangles with the base given by the distance between both masses

$$L4 = (\mu - \tfrac{1}{2}, \tfrac{\sqrt{3}}{2})$$
$$\text{L5}=(\mu - \tfrac{1}{2}, -\tfrac{\sqrt{3}}{2})$$

Stability of L4 and L5

To know the stability, we used the same previous method :

$$\begin{cases} x=\mu - \frac{1}{2} + \epsilon q_1 \\ y=\pm\frac{\sqrt{3}}{2} + \epsilon q_2 \end{cases}$$

By inserting in function f :

$$\begin{cases} \ddot{q_1} = 2\dot{q_2} + \frac{3}{4}q_1 \pm \frac{3\sqrt{3}(1-2\mu)}{4}q_1 \\ \ddot{q_2} = -2\dot{q_1} + \frac{9}{4}q_2 \pm \frac{3\sqrt{3}(1-2\mu)}{4}q_2 \end{cases} \tag{18}$$

Stability of L4 and L5

Let's assume : $q_i = A_i e^{i\Omega t}$

With the previous equation and by using matrix :

$$\begin{aligned} (-\Omega^2 I - 2i\Omega I - W)A = 0 \\ \iff det(-\Omega^2 I - 2i\Omega I - W) = 0 \\ \iff \Omega^4 - \Omega^2 + \frac{27}{16}[1 - (1-2\mu)^2] = 0 \\ \implies \Omega^2 = \frac{1}{2}(1 \pm [1 - \frac{27}{4}(1 - (1-2\mu)^{1/2})]) \end{aligned} \tag{19}$$

Stability condition

To be stable, Ω^2 must be positive and real :

$$1 - \frac{27}{4}(1 - (1 - 2\mu)^2) > 0 \iff \mu < 0.0385 \tag{20}$$

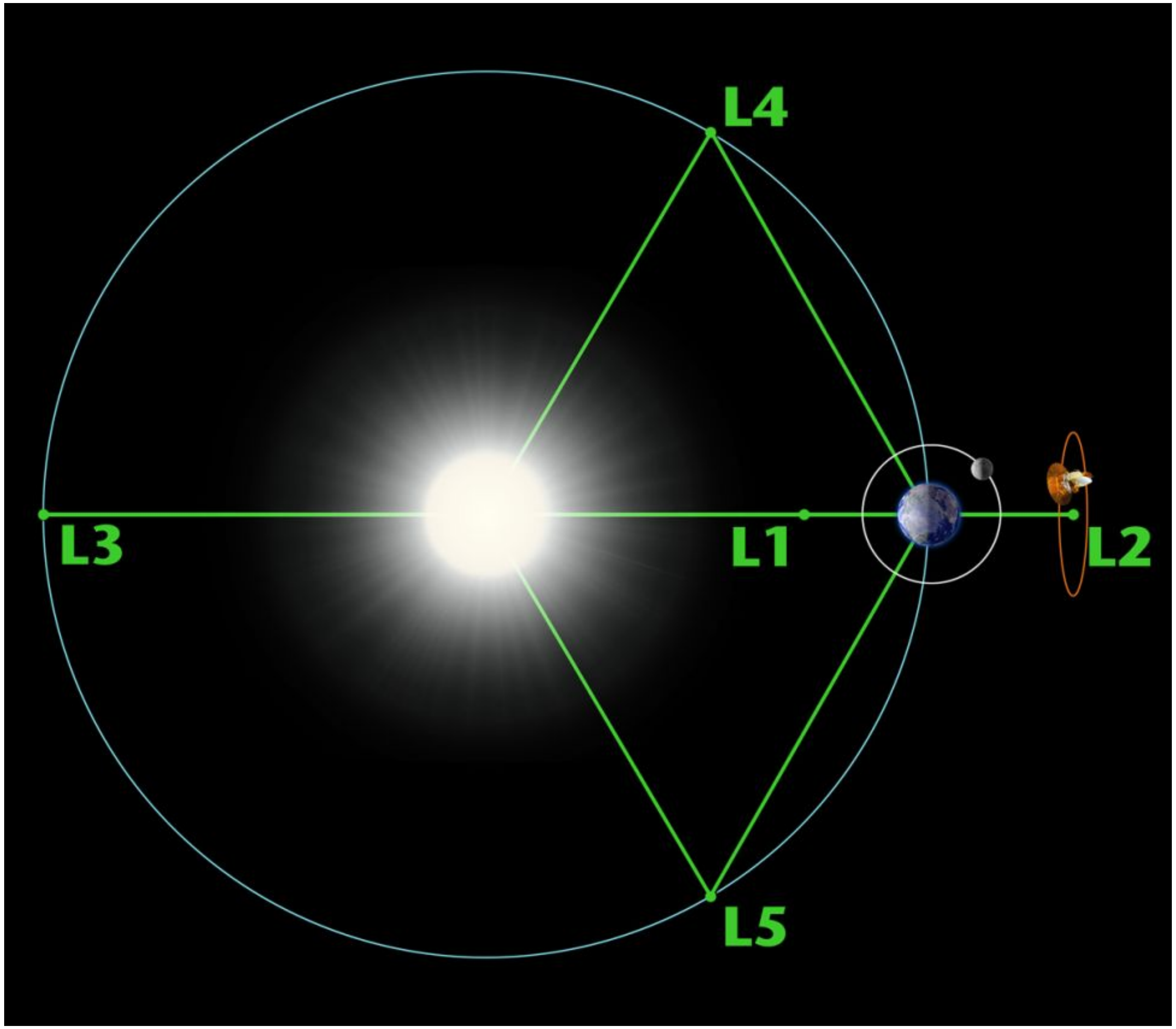


Diagram of the Lagrange points associated with the sun-Earth system.
(Image: © NASA / WMAP Science Team)

Examples of Lagrangian points

The Sun-Earth system

This table gives the approximate calculation formulas for the x and y coordinates of the first three points.

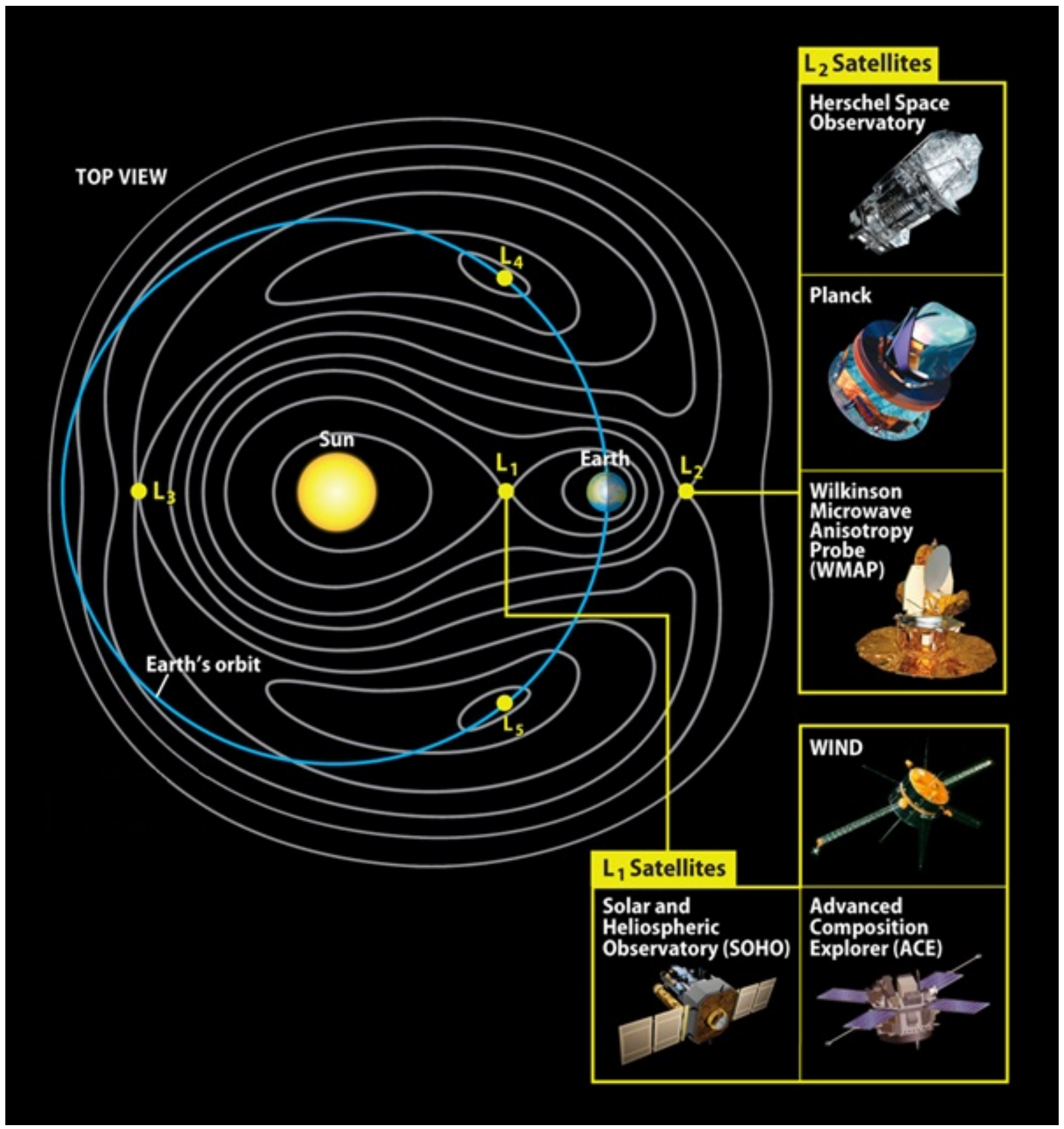


Diagram of the Lagrange points associated with the sun-Earth system.
(Image: © NASA / WMAP Science Team)

$\alpha = \frac{m}{M+m}$ $\beta = \frac{M-m}{M+m}$	$M = 1{,}9889 \times 10^{30} kg$ $m = 5{,}9742 \times 10^{24} kg$ $\alpha = 3 \times 10^{-6}$ $\beta = 0{,}999994$ $R = 1{,}4959 \times 10^{8}\ km$	$M = 5{,}9742 \times 10^{24} kg$ $m = 7{,}36 \times 10^{22} kg$ $\alpha = 0{,}0122$ $\beta = 0{,}976$ $R = 384\ 300\ km$

Points	x	y	Earth-Sun
L_1	$\mathbf{R}\left(1-\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right)$	0	1,4809 10^8 km
L_2	$\mathbf{R}\left(1+\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right)$	0	1,5108 10^8 km
L_3	$-\mathbf{R}\left(1+\frac{5\alpha}{12}\right)$	0	-1,4959 10^8 km

The Sun-Earth system

- Points L4 and L5 lie along Earth's orbit at 60 degrees ahead of and behind Earth, forming the apex of two equilateral triangles.
- Dust and asteroids tend to accumulate in these regions: the 2010 TK7 asteroid.

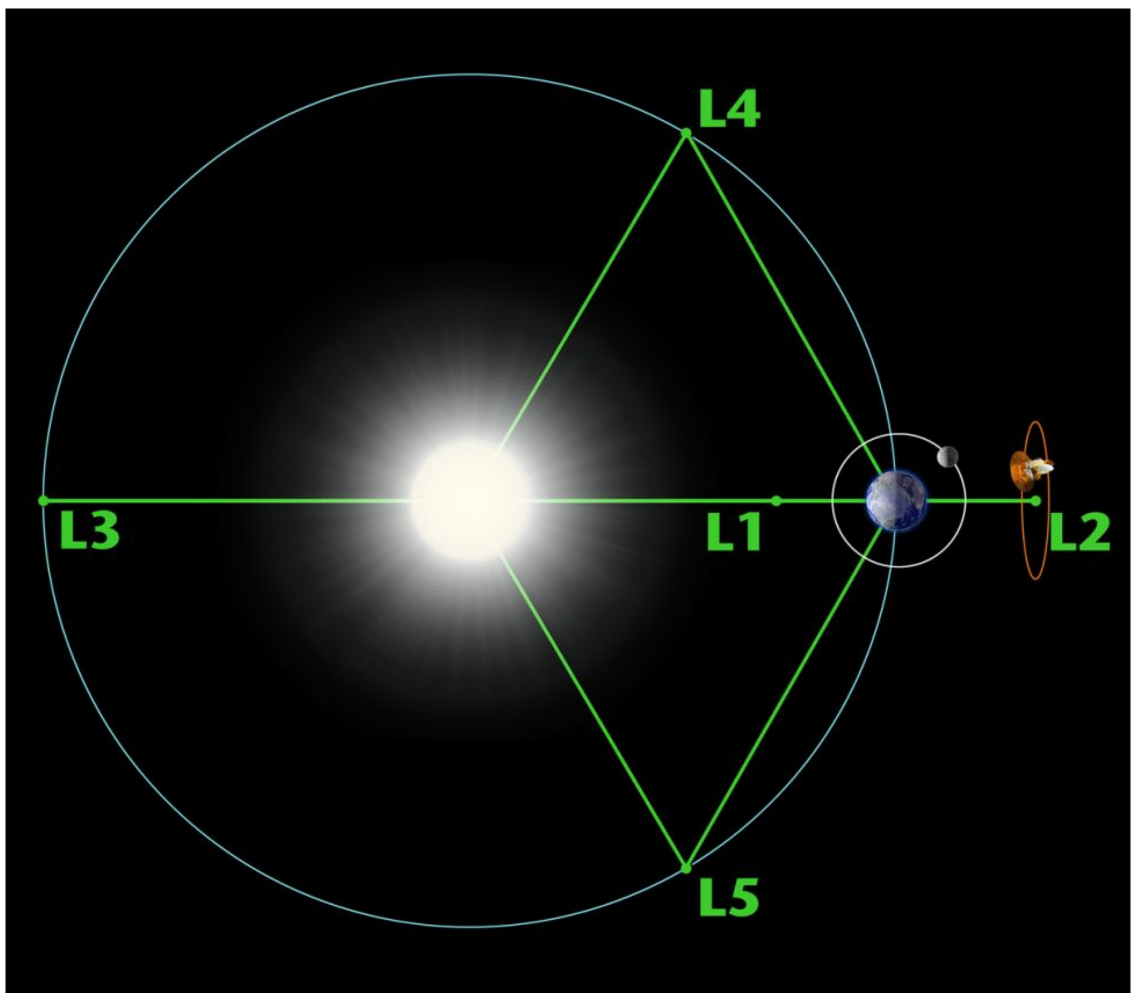
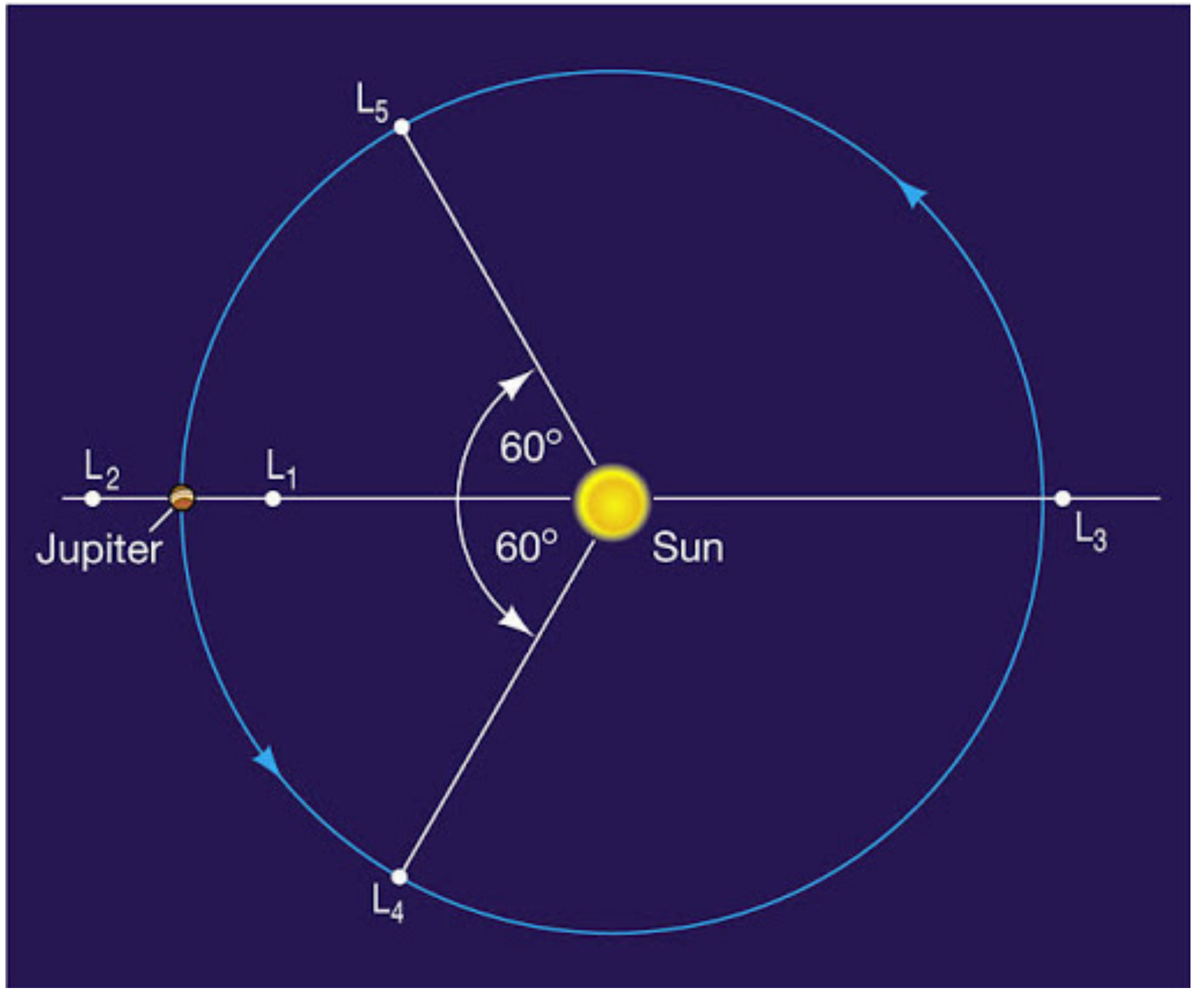


Diagram of the Lagrange points associated with the sun-Earth system.
(Image: © NASA / WMAP Science Team)

$\alpha = \dfrac{m}{M+m}$ $\beta = \dfrac{M-m}{M+m}$	$M = 1{,}9889 \times 10^{30} kg$ $m = 5{,}9742 \times 10^{24} kg$ $\alpha = 3 \times 10^{-6}$ $\beta = 0{,}999994$ $R = 1{,}4959 \times 10^{8}\ km$	$M = 5{,}9742 \times 10^{24} kg$ $m = 7{,}36 \times 10^{22} kg$ $\alpha = 0{,}0122$ $\beta = 0{,}976$ $R = 384\ 300\ km$

Points	x	y	Earth-Sun
L4	$\frac{R}{2}\beta$	$\frac{\sqrt{3}}{2}R$	x = 7,4794 10^7 km y = 1,2955 10^8 km
L5	$\frac{R}{2}\beta$	$-\frac{\sqrt{3}}{2}R$	

The Sun-Jupiter system



http://spaceguard.iaps.inaf.it/NScience/neo/dictionary/lagrange.htm

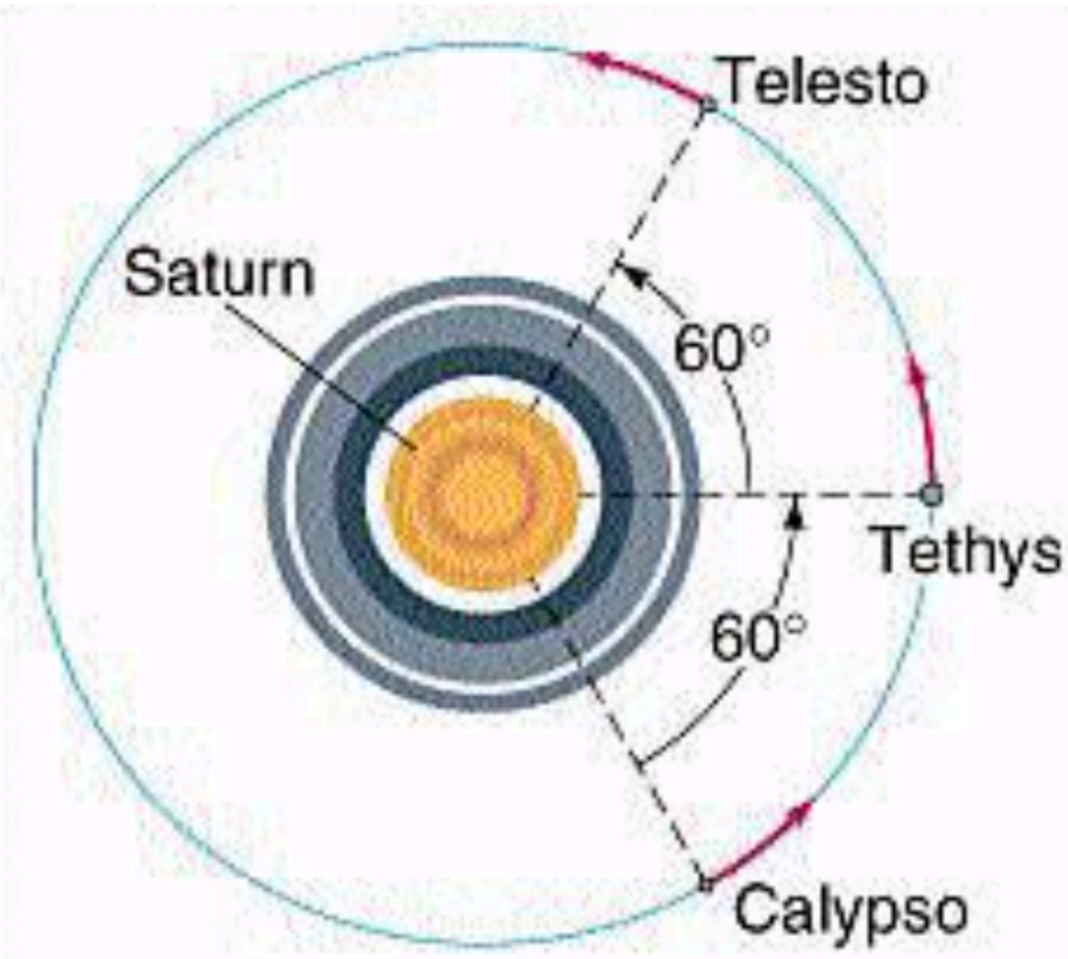
With $M = 1{,}9889.10^{30}$ km
$m = 1{,}898.10^{27}$ km
$R = 7{,}783.10^{8}$ km

Points	x	y	Sun-Jupiter
L_1	$\mathbf{R}\left(\mathbf{1}-\left(\frac{\alpha}{\mathbf{3}}\right)^{\frac{1}{3}}\right)$	$\mathbf{0}$	7.2645×10^{8}km
L_2	$\mathbf{R}\left(\mathbf{1}+\left(\frac{\alpha}{\mathbf{3}}\right)^{\frac{1}{3}}\right)$	$\mathbf{0}$	8.3265×10^{8} km
L_3	$-\mathbf{R}\left(\mathbf{1}+\frac{\mathbf{5}\alpha}{\mathbf{12}}\right)$	$\mathbf{0}$	-7.7791×10^{-8} km
L_4	$\frac{\mathbf{R}}{\mathbf{2}}\beta$	$\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{R}$	$x=3{,}884 \times 10^{8}$ km
L_5	$\frac{\mathbf{R}}{\mathbf{2}}\beta$	$-\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{R}$	$y=6{,}740 \times 10^{8}$ km

- Each of the stable Lagrange points forms an equilateral triangle.
- Points L4 and L5 are stable, asteroids that surround the L4 and L5 points are called Trojans in honor of the story of Troy that are between Jupiter and the Sun.

The Saturn-Tethys moon system

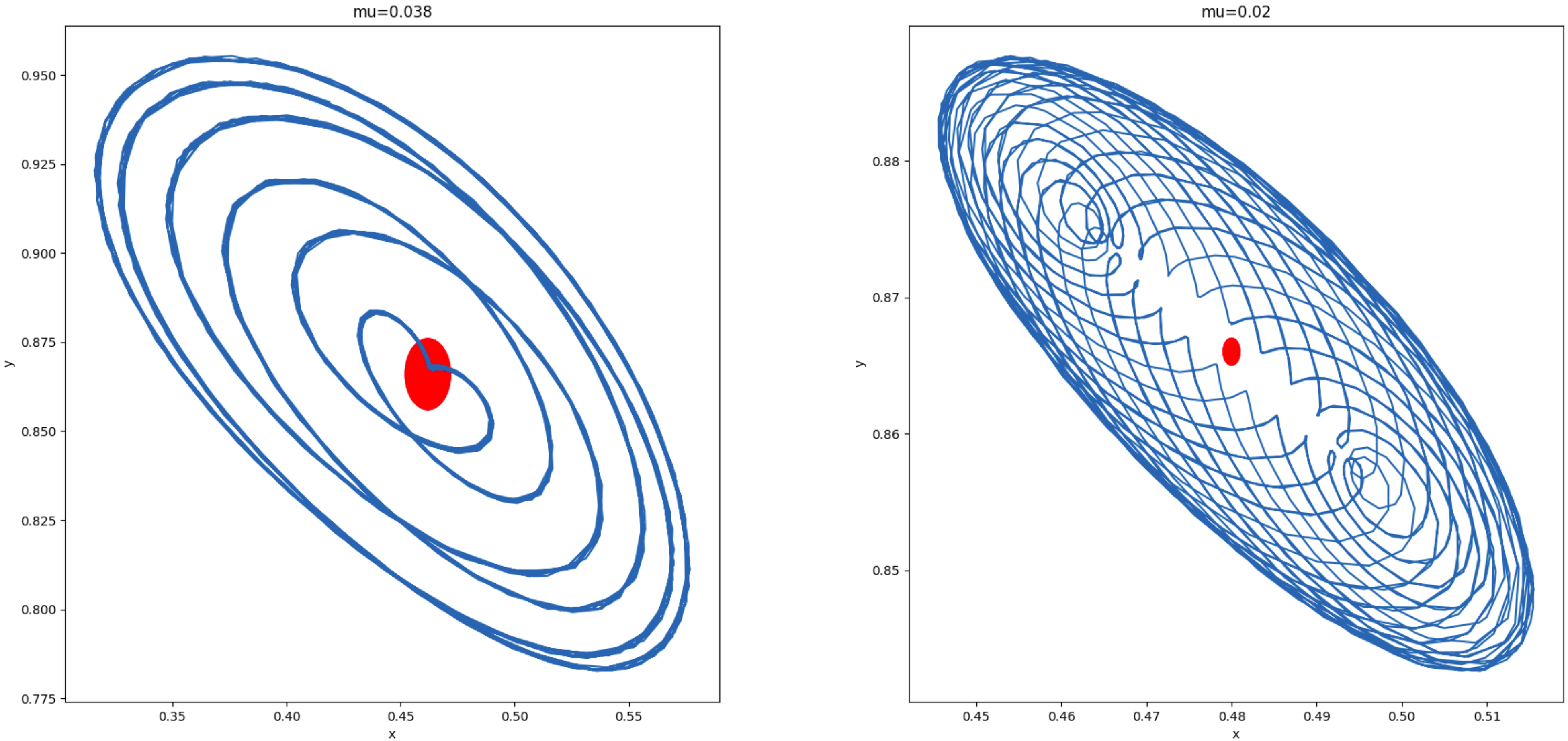
With $M=1,9889.10^{30}$ km
$m = 5,683.10^{26}$ km
R=294 619 km



http://staff.on.br/jlkm/astron2e/AT_MEDIA/CH12/CHAP12AT.HTM

Points	x	y	Saturn-Tethys
L1	$R\left(1-\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right)$	0	281 165 km
L2	$R\left(1+\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right)$	0	308 072 km
L3	$-R\left(1+\frac{5\alpha}{12}\right)$	0	-294 661 km
L4	$\frac{R}{2}\beta$	$\frac{\sqrt{3}}{2}R$	x=147225 km
L5	$\frac{R}{2}\beta$	$-\frac{\sqrt{3}}{2}R$	y=255147 km

- The Saturnian moon Tethys has two smaller moons in its L4 and L5 points, Telesto and Calypso.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0.875
0.850
0.825
0.800
0.775
y
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0.40
0.45
0.50
0.55
x
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0.86
0.85
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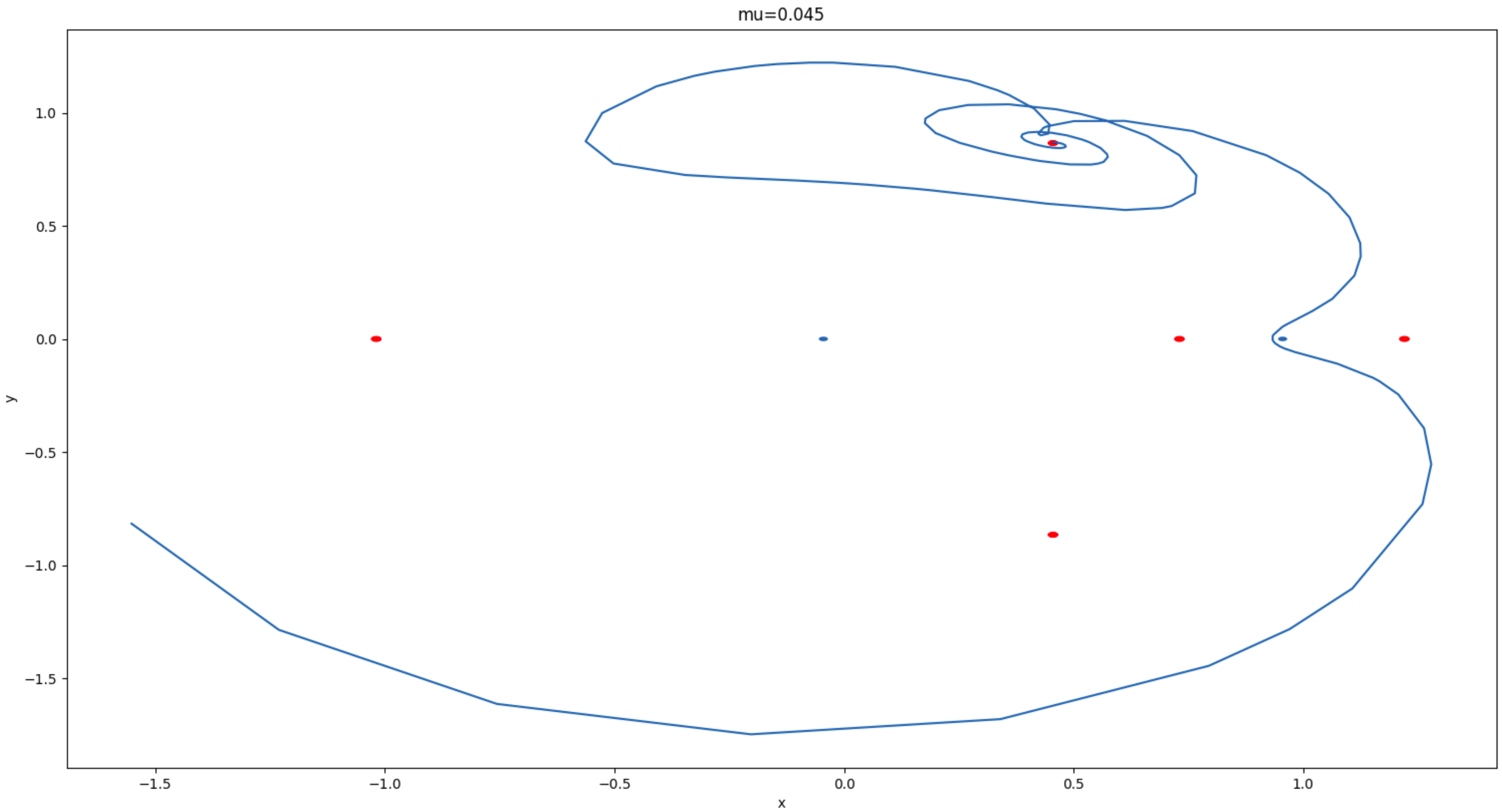
0.46
0.47
0.48
0.49
0.50
0.51
x



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−1.0
−1.5
y
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1.0
x