Linear and Combinatorial Optimization 2020

LECTURE 8

Overview

1 The assignment problem

The assignment problem

Example (p. 251)

Suppose that n people, $P_1, \ldots, P_n$, are being considered for n jobs, $J_1, \ldots, J_n$. Using various criteria, including past performance, aptitude, and job ability, we specify a value c_{ij} for person i to get assigned job j. We assume that each person is assigned exactly one job and that each job is assigned to exactly one person. The problem is to assign the people to the jobs so that the total value of the assignment is maximized.

To construct the mathematical model, define the variables x_{ij} so that

$$x_{ij} = \begin{cases} 1 & \text{if } P_i \text{ is assigned to } J_j, \\ 0 & \text{otherwise.} \end{cases}$$

The assignment problem

Example (Cont.)

Then the mathematical model is

$$\text{maximize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{n} x_{ij} = 1 & \text{for all } i, \\ \sum_{i=1}^{n} x_{ij} = 1 & \text{for all } j, \\ \quad x_{ij} = 0 \text{ or } 1. \end{cases}$$

The assignment problem

- The assignment problem can be solved as an ordinary LP problem (by the Kruskaal–Hoffman theorem),

- or with the transportation algorithm after noticing that the assignment problem is a transportation problem with demand and supply vectors only containing ones,

- or with the so called Hungarian method (named in honour of the mathematicians König and Egerváry who provided the base for it) that we will learn now.

- We will assume that the problem is phrased as a *minimization problem* with a cost matrix with nonnegative entries. We will see later how we will convert a maximization problem to this situation.

The Hungarian method

- Clearly, a possible way to assign the jobs is to give job number j to person number j.
- For any permutation of the numbers $1, \ldots, n$, we get a new way of assigning the jobs.
- A permutation can be represented by a permutation matrix, which has the property that in each row and column, exactly one entry is 1, and the rest are 0. For example, the permutation $(1, 2, 3, 4) \mapsto (4, 2, 1, 3)$ (with the interpretation that job 1 is assigned to person 4, etc.) can be represented by the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that if me multiply the vector $(1, 2, 3, 4)$ from the right by this matrix, we obtain the vector $(4, 2, 1, 3)$.

The Hungarian algorithm, Step 1

The following theorem is the basis for the Hungarian algorithm:

Theorem

If the cost matrix for an $n \times n$ assignment problem has nonnegative entries and at least n zeros, then an optimal solution (with value 0) exists if n of the zeros lie in the positions of the ones of some $n \times n$ permutation matrix $\mathbf{P}$. In this case, the matrix $\mathbf{P}$ represents an optimal assignment.

Proof.

In the situation described, the cost can never be smaller than zero, and we have found an assignment for which the value is 0. □

The Hungarian method, Step 1

- The situation of the theorem above may look very unlikely, and we shouldn't expect that the assumptions of the theorem are valid for the cost matrix **C**.

- Never-the-less, the theorem is the basis for an algorithm, since we may be able to modify the cost matrix in such a way that the optimal solution doesn't change and the new cost matrix satisfies the conditions of the theorem.

- The next theorem gives a hint on how we could change the cost matrix **C**.

Theorem

Suppose that the matrix $\mathbf{C} = [c_{ij}]$ is the cost matrix for an $n \times n$ assignment problem. Suppose that $\widehat{\mathbf{X}} = [\widehat{x}_{ij}]$ is an optimal solution to this problem. Let $\mathbf{C}'$ be the matrix formed by adding the number α to each entry of one of the rows (or column) of $\mathbf{C}$. Then $\widehat{\mathbf{X}}$ is an optimal solution of the new assignment problem defined by $\mathbf{C}'$.

Proof.

We assume that α is added to row number r. The objective function for the new problem is

$$
\begin{aligned}
z' &= \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} x_{ij} \\
&= \sum_{\substack{i=1 \\ i \neq r}}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} (c_{rj} + \alpha) x_{rj} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} + \alpha \sum_{j=1}^{n} x_{rj} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} + \alpha.
\end{aligned}
$$

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The Hungarian method, Step 1

Proof (Cont).

(Can you see what property of **X** that was used in the last step?) The computation above show that the smallest value for z' will be obtained when

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

is the smallest possible, i.e. when $\mathbf{X} = \widehat{\mathbf{X}}$.
The proof is similar if the number α is instead added to all entries in a column. $\qquad\square$

With this theorem in mind, the strategy for the algorithm is to modify **C** by adding constants to rows or columns until the previous theorem can be used.

The Hungarian method, Step 1

Example

Consider the assignment problem with cost matrix

$$\mathbf{C} = \begin{bmatrix} 4 & 2 & 9 & 7 \\ 7 & 8 & 5 & 6 \\ 3 & 3 & 4 & 1 \\ 7 & 5 & 2 & 6 \end{bmatrix}.$$

The optimal solution of the assignment problem is not changed if we subtract the lowest number of each row of C. We obtain the new matrix

$$\mathbf{C_1} = \begin{bmatrix} 2 & 0 & 7 & 5 \\ 2 & 3 & 0 & 1 \\ 2 & 2 & 3 & 0 \\ 5 & 3 & 0 & 4 \end{bmatrix}.$$

The Hungarian method, Step 1

Example (Cont.)

There is no zero in column 1, but we can replace $\mathbf{C_1}$ by the matrix $\mathbf{C_2}$ which is obtained by subtracting 2 from that column. It is equivalent to solve the assignment problem with cost matrix

$$\mathbf{C_2} = \begin{bmatrix} 0 & 0 & 7 & 5 \\ 0 & 3 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 3 & 3 & 0 & 4 \end{bmatrix}.$$

Perhaps you can already find a permutation matrix which solves the problem? Hint: Look at the last row and column first (which have only one zero each)

The Hungarian method, Step 2

Example (Cont.)

- *Despite the fact that we already know an optimal solution, we will solve the problem systematically, in order to understand the algorithm.*

- *Instead of filling in 0's and 1's in a separate permutation matrix, we will star certain 0 entries in the cost matrix (where the corresponding entry of the permutation matrix is set to 1). When doing so, we say that a zero (of the cost matrix) has been assigned.*

- *We start in the first row and assign the first possible zero (reading from left to right in that row) which belongs to a column without previously assigned zeros.*

The Hungarian method, Step 2

Example (Cont.)

■ *Following this procedure, we can assign zeros in the first three rows, but not in the fourth row. The fourth person has not been assigned a job.*

$$\begin{bmatrix} 0^* & 0 & 7 & 5 \\ 0 & 3 & 0^* & 1 \\ 0 & 2 & 3 & 0^* \\ 3 & 3 & 0 & 4 \end{bmatrix}.$$

■ *There can be two explanations for this:*
  ■ *Either it is not possible to assign zeros with exactly one zero being assigned in each row and column, or*
  ■ *it is possible to assign zeros, but the algorithm has not found the correct solution. (Actually, we know that we are in this situation, but usually we will not be sure).*

The Hungarian method, Step 3

Example (Cont.)

- *We try to reassign the zeros so that each row and column has exactly one assigned zero.*

- *Try to find a chain of alternating 0's and 0^*'s starting from a zero in the first row in which we didn't manage to assign a zero and continuing the sequence as long as possible. In our example, we get (starting from the last row):*

$$\begin{bmatrix} 0^* & 0 & 7 & 5 \\ 0 & 3 & 0^* & 1 \\ 0 & 2 & 3 & 0^* \\ 3 & 3 & 0 & 4 \end{bmatrix}$$

For the sequence, switch 0 to 0^* and 0^* to 0.

The Hungarian method, Step 3

Example

■ *The resulting matrix is*

$$\begin{bmatrix} 0 & 0^* & 7 & 5 \\ 0^* & 3 & 0 & 1 \\ 0 & 2 & 3 & 0^* \\ 3 & 3 & 0^* & 4 \end{bmatrix}.$$

■ *Note that the assigned zeros correspond to the ones in a permutation matrix, and so we have found an optimal assignment, where job* 1 *is assigned to person number* 2, *job* 2 *is assigned to person number* 1, *job* 3 *is assigned to person number* 4 *and job* 4 *is assigned to person number* 3.

The Hungarian method, Step 3

- Now, we give the details how to construct the paths of alternating 0's and 0^*'s.

- Suppose we have reached a 0 in position (i, j). We search a 0^* in column j. Note that there can be at most one.

- If a 0^* is found, we add it to the sequence.

- If there is no 0^* in column j, we end the sequence here, and change the 0's to 0^*'s and vice versa for the whole sequence.

The Hungarian method, Step 3

- Suppose that we have reached a 0^* in position (i, j). We search row i for a 0 that does not lie in a column appearing previously in the path.

- Note that there can be more than one, so we should do this systematically and take the first possible one (from the left).

- If there are none, we label the column of the last 0^* that we found as *necessary*, and then we backtrack the path, i.e. we delete the last $0*$ and 0 and look for another 0 two steps earlier in the sequence. Take the next possible 0 from the left in the row of the now last 0^* of the sequence. If there are none, then we label also the column of the current 0^* as necessary, and delete one more pair of 0^*'s and 0's, and take the next possible 0 if there is one. Etc.

The Hungarian method, Step 3

- The construction of the sequence can end in two different ways:
- Either we managed to change 0* to 0 for the whole sequence, and we have found an optimal assignment, or
- All the 0's and 0*'s where deleted of the whole sequence, and so we couldn't reassign the 0's. In this case, we should go to the next step in the algorithm.
- Before studying the next step, we will see an example of the second case, where a reassignment of zeros cannot me made.

The Hungarian method, Step 1–3

Example

Consider the assignment problem with cost matrix

$$\mathbf{C} = \begin{bmatrix} 8 & 7 & 9 & 9 \\ 5 & 2 & 7 & 8 \\ 6 & 1 & 4 & 9 \\ 2 & 3 & 2 & 6 \end{bmatrix}.$$

Construct another cost matrix $\mathbf{C_1}$ by subtracting the smallest entry of each row to all the entries of that row, and then doing the same for each column. We obtain

$$\mathbf{C_1} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 5 & 1 \\ 5 & 0 & 3 & 6 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

The Hungarian method, Step 1–3

Example (Cont.)

Next, we start assigning 0's starting in the first row and going from left to right. We get the matrix

$$\begin{bmatrix} 1 & 0^* & 2 & 0 \\ 3 & 0 & 5 & 1 \\ 5 & 0 & 3 & 6 \\ 0^* & 1 & 0 & 2 \end{bmatrix}$$

where we managed to assign 0's only in row 1 and 4.

As row 2 is the first row without an assigned zero, we start constructing a sequence starting from the 0 in position $(2,2)$. We search for a 0^* in column 2 and find one in position $(1,2)$. Then we search for a 0 in row 1 and find one in position $(1,4)$. As we cannot find a 0^* in column 4, we end the sequence here, and interchange 0 with 0^* and vice versa in the whole sequence.

Example (Cont.)

The resulting matrix is given by

$$\begin{bmatrix} 1 & 0 & 2 & 0^* \\ 3 & 0^* & 5 & 1 \\ 5 & 0 & 3 & 6 \\ 0^* & 1 & 0 & 2 \end{bmatrix}$$

and the result was that we managed to assign one more zero than we had before. Next, we try to do the same thing one more time starting from position $(3, 2)$, where we find a zero in row 3. We look for a 0^* in column 2 and find one in position $(2, 2)$, but we fail to find a 0 in row 2. We label column 2 as necessary. Then we delete the 0^* and 0 of the sequence and try to find another 0 in row 3, but this is not possible, since row 3 has only one 0. In this situation, we cannot make a complete assignment of zeros.

The Hungarian method, Step 4

Next, we are thinking a bit differently about the problem, and consider how many lines are needed to cover the 0's of a square matrix. (Actually, this has something to do with the dual problem). Note that

- Any pattern of zeros in an $n \times n$ matrix can be covered by (at most) n lines. This is obvious, since we can cover all the entries by letting each line cover one column.

The number of lines that are needed to cover the zeros can be found by the next theorem:

Theorem (König)

The maximum number of zeros that can be assigned is equal to the minimum number of lines that are needed to cover all the zeros.

The Hungarian method, Step 4

- Suppose that the zeros in the $n \times n$ matrix **C** can be covered by k lines, where $k < n$. Note that by the above theorem, this will be the case if we have reached this stage of the algorithm.

- The lines can be found from the information of step 3 of the algorithm: We had labelled a column as *necessary*, if we deleted (backtracked) a 0^* in that column. We now also label a row as *necessary* if it contains a 0^* in an unnecessary column.

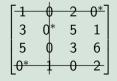- Cover all the necessary rows and columns. This will automatically cover all the zeros of **C**.

The Hungarian method, Step 4

Example

In our last example, we had labelled column 2 as necessary, and all other columns are unnecessary. The first row has a 0^* in column 4, and so the first row is necessary. Row 2 and 3 are not necessary since row 2 contains a 0^* but it is in a necessary column, and row 3 doesn't contain a 0^*. Row 4 is necessary since it contains a 0^* in column 1 (which is unnecessary). When covering the necessary rows and columns, we get the following picture.

$$\begin{bmatrix} 1 & 0 & 2 & 0^* \\ 3 & 0^* & 5 & 1 \\ 5 & 0 & 3 & 6 \\ 0^* & 1 & 0 & 2 \end{bmatrix}$$

Note that all the zeros (both 0 and 0^*) have been covered by lines.
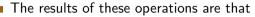
The Hungarian method, Step 4

- Let a be the smallest of the uncovered entries of **C**.
- We form a new matrix **C**′ by
  - subtracting a from the entries of every *uncovered row*, and
  - adding a to the entries of each *covered column*.
- Note that the optimal solution of **C** and **C**′ is the same after these operations, by the theorem in Step 1.

- The results of these operations are that
  - each uncovered entry of a has decreased by a (since it belongs to an uncovered row and an uncovered column),
  - each entry covered by exactly one line has remained unchanged (either it belongs to a covered row and an uncovered column and was not modified, or it belongs to an uncovered row and an uncovered column and got a added *and* subtracted to it).
  - each entry covered by two lines has increased by a (since it belongs to a covered row and a covered column).

- The rules for obtaining $\mathbf{C}'$ from $\mathbf{C}$ can be more easily stated: Add a to each entry that is covered by 2 lines and subtract a to each entry that is not covered. Leave the rest of the entries as they are.

- After these step, we have created at least one more zero, and can go back to Step 2 of the algorithm.

Example (Cont.)

The smallest of the uncovered entries of our matrix is 1 (see left matrix below). After we apply the rules to this matrix (with $a = 1$), we obtain the matrix to the right.

$$
\begin{bmatrix}
1 & 0 & 2 & 0^* \\
3 & 0^* & 5 & 1 \\
5 & 0 & 3 & 6 \\
0^* & 1 & 0 & 2
\end{bmatrix}
\quad \mapsto \quad
\begin{bmatrix}
1 & 1 & 2 & 0^* \\
2 & 0^* & 4 & 0 \\
4 & 0 & 2 & 5 \\
0^* & 2 & 0 & 2
\end{bmatrix}
$$

We have got a new matrix, and can go back to Step 2 of the method.

The Hungarian method

Example

We start over with the matrix (so all the 0^* are first changed to 0), and then we start assigning zeros in the cost matrix, with the following result:

$$\begin{bmatrix} 1 & 1 & 2 & 0^* \\ 2 & 0^* & 4 & 0 \\ 4 & 0 & 2 & 5 \\ 0^* & 2 & 0 & 2 \end{bmatrix}$$

As we didn't manage to assign zeros in all four rows, we go to Step 3 and try to reassign the zeros. We get a sequence of positions
$(3, 2) \rightarrow (2, 2) \rightarrow (2, 4) \rightarrow (1, 4)$, *but there is no 0 in row 1, and so we cannot continue the sequence. The fourth column is labelled as necessary.*
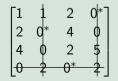
The Hungarian method

Example (Cont.)

Next, we backtrack the sequence to the 0^* at position $(2, 2)$. We look for another 0 in row 2, but there are none. So column 2 is marked as necessary too, and we backtrack to the third row, trying to find another 0 to start with. As there are none, we need to continue to Step 4.

We have found two necessary columns: 2 and 4. Next we label the rows with a 0^* in an unnecessary column as necessary, and find that the only necessary row is row 4.

Covering the necessary rows and columns with lines, we get the figure

$$\begin{bmatrix} 1 & 1 & 2 & 0^* \\ 2 & 0^* & 4 & 0 \\ 4 & 0 & 2 & 5 \\ 0 & 2 & 0^* & 2 \end{bmatrix}$$

The Hungarian method

Example (Cont.)

Noticing that the smallest uncovered entry is 1*, we add* 1 *to the doubly covered entries and subtract* 1 *from the uncovered ones. We get the new matrix*

$$
\begin{bmatrix}
0 & 1 & 1 & 0^* \\
1 & 0^* & 3 & 0 \\
3 & 0 & 1 & 5 \\
0 & 3 & 0^* & 3
\end{bmatrix}
$$

One more zero was created, and we can start over from Step 2*. After removing all the lines and replacing* 0^* *by* 0*, we start reassigning zeros.*

The Hungarian method

Example (Cont.)

We get

$$
\begin{bmatrix}
0^* & 1 & 1 & 0 \\
1 & 0^* & 3 & 0 \\
3 & 0 & 1 & 5 \\
0 & 3 & 0^* & 3
\end{bmatrix}
$$

There is no 0^* in row 3, and we move to Step 3. We find a sequence of alternating 0's and 0^*'s starting at position $(3, 2)$: It is $(3, 2) \rightarrow (2, 2) \rightarrow (2, 4)$. The sequence stops at $(2, 4)$ since there is no 0^* in column 4. When 0 is changed to 0^* and vice versa in the sequence, we have found a complete assignment.

The Hungarian method

Example (Cont.)

The matrix with the completely assigned zeros is given by

$$\begin{bmatrix} 0^* & 1 & 1 & 0 \\ 1 & 0 & 3 & 0^* \\ 3 & 0^* & 1 & 5 \\ 0 & 3 & 0^* & 3 \end{bmatrix}$$

We have found an optimal solution: Person 1 takes job number 1, person 2 takes job number 4, person 3 takes job number 2 and person 4 takes job number 3.

The Hungarian method

- The method that was described was for solving a minimization problem. If we are instead given a maximization problem, as the originally stated assignment problem, we can convert it to a minimization by replacing the value matrix **C** by its negation $-$**C**.

- If some of the entries of **C** were positive, these are now negative, and so we still cannot apply the method directly, but recall the theorem that the optimal solution doesn't change if we add any constant number to each entry in a row or column.

- Hence, by adding to each row, the absolute value of the most negative entry to that row we get a cost matrix on which we can apply the method.

The Hungarian method

Example

Suppose an assignment problem asks one to maximize the total value of the assignment for which the individual values are given by

$$\mathbf{C} = \begin{bmatrix} 3 & 7 & 4 & 6 \\ 5 & 2 & 8 & 5 \\ 1 & 3 & 4 & 7 \\ 6 & 5 & 2 & 6 \end{bmatrix}$$

We convert it to a minimization problem with cost matrix $-\mathbf{C}$, and then add to all the entries of each row the absolute value of the most negative entry of that row.

The Hungarian method

Example (Cont.)

We get the new matrix

$$\mathbf{C} = \begin{bmatrix} 4 & 0 & 3 & 1 \\ 3 & 6 & 0 & 3 \\ 6 & 4 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

for which the Hungarian method can be applied.

The original matrix of the above example is from Example 5 on p. 338 of the book (K–B), although I modified the cost matrix in a slightly different way. My method gives more zeros of the modified cost matrix, and so it is likely that it is possible to solve the assignment problem in fewer steps by using this conversion method.