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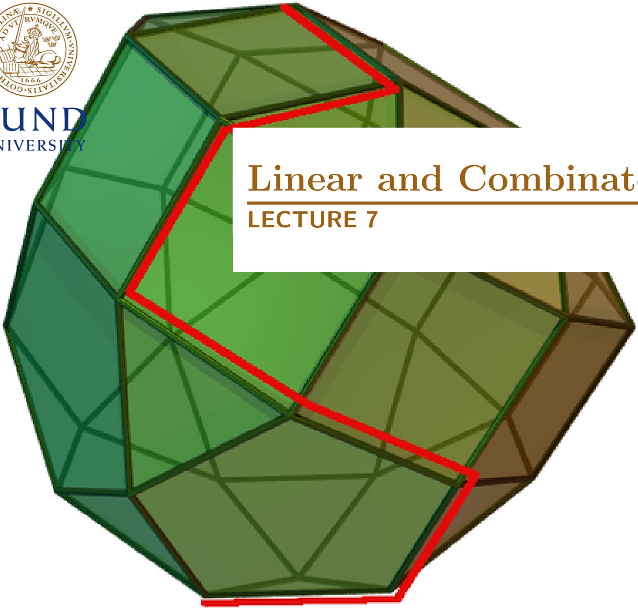

Linear and Combinatorial Optimization 2020

LECTURE 7

Overview

The transportation problem

The transportation problem concerns minimization of the transportation of goods from a company's factories to its warehouses. We start with an example with 3 factories and 4 warehouses.

Example (p. 296)

Let the cost of transportation per unit of goods from factory i to warehouse j be given by the table

	W_1	W_2	W_3	W_4
F_1	5	7	9	6
F_2	6	7	10	15
F_3	7	6	8	1

The transportation problem

Example (Cont.)

- *Let the supply and demand vectors be given by*

$$\mathbf{s} = \begin{bmatrix} 120 \\ 140 \\ 100 \end{bmatrix} \quad \textit{and} \quad \mathbf{d} = \begin{bmatrix} 100 \\ 60 \\ 80 \\ 120 \end{bmatrix}, \quad \textit{respectively.}$$

They represent the amount of the product that each factory can supply and each warehouse demands, respectively.

- *The problem is feasible if and only if the total supply is not less than the total demand, i.e. if and only if* $\sum_{i=1}^{3} s_i \geq \sum_{j=1}^{4} d_j$. *We have* $\sum_{i=1}^{3} s_i = 120 + 140 + 100 = 360$ *and* $\sum_{j=1}^{4} d_j = 100 + 60 + 80 + 120 = 360$, *so the feasibility condition is satisfied.*

The transportation problem

- In the theory for the transportation problem, it can always be assumed that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ (for feasible problems), since if $\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$, then we can introduce an extra column in the table for transportation costs, that represent the amount of goods that stays in the factory. We let the shipping costs be 0 for that column. We also need to add an extra entry to the demand vector with the missing amount, so that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.
- From now on, we will assume that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$.

The transportation problem

- The problem that we need to solve is

$$
\begin{array}{ll}
\text{minimize} & z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\
\text{subject to} & \begin{cases} \sum_{j=1}^{n} x_{ij} \leq s_i & \text{for all } i, \\ \sum_{i=1}^{m} x_{ij} \geq d_j & \text{for all } j, \\ x_{ij} \geq 0, & \text{integers.} \end{cases}
\end{array}
$$

- Since we have assumed that $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$, it follows that the $\leq$ and $\geq$ signs for the supply and the demand constraints can be replaced by equalities. (See next slide for a proof of this.)

The transportation problem

- **Proof that $\sum_{j=1}^{n} x_{ij} \leq s_i$ can be replaced by $\sum_{j=1}^{n} x_{ij} = s_i$:**
  If the actual supply $\sum_{j=1}^{n} x_{ij}$ would be strictly less than s_i, then the total amount shipped would be $\sum_{i,j} x_{ij} < \sum_{i=1}^{m} s_i = d_j$, so that the total amount shipped would be less than the total demand. This is clearly a contradiction.
- Therefore, we solve

$$
\begin{array}{ll}
\text{minimize} & z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \\
\text{subject to} & \begin{cases} \sum_{j=1}^{n} x_{ij} = s_i & \text{for all } i, \\ \sum_{i=1}^{m} x_{ij} = d_j & \text{for all } j, \\ x_{ij} \geq 0, & \text{integers.} \end{cases}
\end{array}
$$

The transportation problem

- Since $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$, one of the constraints is redundant, and so we can remove any one of them.
- To see that the last constraint can be removed (for example), we add all the supply constraints and obtain (since we can change the order of summation)

$$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Introduce all the demand constraints except the last one, and sum them:

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} d_j.$$

Subtracting these two equations, we obtain

$$\sum_{i=1}^{m} x_{in} = d_n,$$

which is the last constraint.

The transportation problem

- If we would like to use the simplex method to solve the transportation problem, we first remove one of the constraints. Then we will get an integer optimal solution automatically (by the Kruskaal–Hoffman theorem).
- The transport algorithm is based on the simplex method, but takes advantage of the special structure of the transportation problem.
- We will now learn the transportation algorithm, starting from our previous concrete example.

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Example (Cont.)

Fill in the table in the following way: Start by writing the costs in each corner of the boxes:

	W_1	W_2	W_3	W_4	
F_1	5	7	9	6	120
F_2	6	7	10	5	140
F_3	7	6	8	1	100

The transportation problem

Example (Cont.)

- *We start by filling in the table with amounts to find an initial basic feasible solution. There are different rules that one can use for this. Right now we will use the simplest one, the* northwest rule*. We will see some other rules later today.*
- *Start in the top left corner of the table, and fill in as much as possible while stepping right and down until all the constraints are met.*
- *Put zeros in the rest of the boxes.*
- *This procedure always gives a basic feasible solution.*
- *The resulting table can be seen in the next slide.*

The transportation problem

Example (Cont.)

	W_1	W_2	W_3	W_4	
F_1	cost 5, 100	cost 7, 20	cost 9, 0	cost 6, 0	120
F_2	cost 6, 0	cost 7, 40	cost 10, 80	cost 5, 20	140
F_3	cost 7, 0	cost 6, 0	cost 8, 0	cost 1, 100	100
	100	60	80	120	

The transportation problem

Example (Cont.)

- *Now, we would like to modify the table. The current transportation cost is* $500 + 140 + 280 + 800 + 100 + 100 = 1920$. *Is this optimal? Probably not, but we will soon see.*
- *Try to modify the table. What would happen if we increase one of the unused routes from* 0 *to* 1 *and modify other* used *routes so that feasibility is preserved?*
- *In the next slide, we show what happens if* x_{21} *is changed from* 0 *to* 1.
- *Remember that we are only allowed to change nonzero numbers (except the first one).*

The transportation problem

Example (Cont.)

	W_1	W_2	W_3	W_4	
F_1	5 / 99	7 / 21	9 / 0	6 / 0	120
F_2	6 / 1	7 / 39	10 / 80	5 / 20	140
F_3	7 / 0	6 / 0	8 / 0	1 / 100	100
	100	60	80	120	

The transportation problem

Example (Cont.)

- *The total transportation cost has changed from* 1920 *to* $1920 - 5 + 7 - 7 + 6 = 1921$.
- *Since the cost became higher, it was not a good idea to change that particular route.*
- *How do we know how to change the table so that the cost will decrease? We will need some theory for this.*

- We can use the dual problem for this. It is given by

$$\text{maximize} \quad \sum_{i=1}^{m} v_i + \sum_{j=1}^{n} w_j,$$

$$\text{subject to} \begin{cases} v_i + w_j \leq c_{ij} & i_1, \ldots, m, j = 1, \ldots, n, \\ v_i, w_j \in \mathbb{R} & \text{(unrestricted).} \end{cases}$$

The transportation probem

Definition

The reduced costs are $\widehat{c}_{ij} = c_{ij} - v_i - w_j$

Note that the reduced costs would appear as slack variables when solving the dual system with the simplex method.

- Use complementary slackness to find a solution to the dual system. This solution will be infeasible if the solution of the primal problem is not optimal. We need to solve

$$\widehat{c}_{ij} x_{ij} = 0 \qquad \text{for all } i, j.$$

- We conclude that $\widehat{c}_{ij} = 0$ if $x_{ij} \neq 0$. In the next slide, we will try this out for our example.

The transportation problem

Example (Cont.)

- *The complementatary slackness condition tells us which $\widehat{c}_{ij}$ have to be 0:*

0	0		
	0	0	0
			0

For these (i, j), we have $c_{ij} = v_i + w_j$.

- *There are six such equations (since there are six zeros in the table), but we have 7 variables, $v_1, \ldots, v_3$ and $w_1, \ldots, w_4$. This originates in the redundancy in the primal system.*
- *This does not cause any problem. We choose a value for one of the variables, i.e. $v_1 = 0$. Then we solve for the other six variables: We get $v_1 = 0$, $v_2 = 0$, $v_3 = -4$, $w_1 = 5$, $w_2 = 7$, $w_3 = 10$, $w_4 = 5$.*

The transportation problem

Example

- *Now that all the v_i and w_j have been found, we can compute the rest of the $\widehat{c}_{ij}$, using $\widehat{c}_{ij} = c_{ij} - v_i - v_j$.*
- *The table for $\widehat{c}_{ij}$ can be completed:*

0	0	−1	1
1	0	0	0
6	3	2	0

- *The interpretation of the reduced cost $\widehat{c}_{ij}$ is the change of the cost if that variable would increase by 1 (while adapting the other variables to that the constraints still hold).*
- *For example, if the x_{21} variable would increase by 1, then the transportation cost would increase by 1, just as we saw earlier.*

The transportation problem

Example (Cont.)

- *With this interpretation, it is clear which route to change. We need to choose one with a negative reduced cost.*
- *There is only one negative reduced cost, and so we change* x_{13} *from* 0 *to* 1*. Then the transportation table will be as on the next slide.*

The transportation problem

Example (Cont.)

	W_1	W_2	W_3	W_4	
F_1	5 / 100	7 / 19	9 / 1	6 / 0	120
F_2	6 / 0	7 / 41	10 / 79	5 / 20	140
F_3	7 / 0	6 / 0	8 / 0	1 / 100	100
	100	60	80	120	

- *We should have changed* x_{13} *with* 20 *instead of just* 1*. Let's do this instead!*

The transportation problem

Example (Cont.)

Each cell shows the cost, then the amount shipped.

	W_1	W_2	W_3	W_4	
F_1	5 / 100	7 / 0	9 / 20	6 / 0	120
F_2	6 / 0	7 / 60	10 / 60	5 / 20	140
F_3	7 / 0	6 / 0	8 / 0	1 / 100	100
	100	60	80	120	

- *The new cost is* $1920 - 1 \cdot 20 = 1900$, *which is smaller.*

The transportation problem

Example (Cont.)

- *This was one step in the transportation algorithm.*
- *Next, compute new reduced costs and choose a new incoming variable.*
- *Repeat until all reduced costs are nonnegative. Then we have reached an optimum.*

Initialization

- There are more efficient initialization rules than the northwest rule that we have seen.
- If we choose an initial basic feasible solution which is close to the optimal one, we may not have to perform so many iterations in the actual algorithm, and hence we may be able to solve the problem faster.
- We will describe two such rules: *The minimal cost rule* and *Vogel's method*.

The minimal cost rule

In the minimal cost rule, we examine each factory in turn, and try to transport as cheaply as possible from that factory.
For our example, we will get the following initial transport scheme which has cost 2160.

	W_1	W_2	W_3	W_4	
F_1	5	7	9	6	
	100	0	0	20	120
F_2	6	7	10	5	
	0	40	0	100	140
F_3	7	6	8	1	
	0	20	80	0	100
	100	60	80	120	

Vogel's method

1. For each row and column of the cost matrix, find the difference between the smallest and the next smallest entry.
2. Select the row or column with the largest difference.
3. Allocate as much as possible for this route. If the row's supply (column's demand), then disregard this row (column) from further consideration.
4. Repeat from 1 until the table has been filled.

Vogel's method

The largest difference is 5 and can be found in the third row, therefore we fill that row first. Recalculating the differences after the third row has been filled, gives that the differences are all 1, except for the second column, which has difference 0. We choose (arbitrarily) to fill in the second row before the first.

	W_1	W_2	W_3	W_4	
F_1	5 / 0	7 / 40	9 / 80	6 / 0	120
F_2	6 / 100	7 / 20	10 / 0	5 / 20	140
F_3	7 / 0	6 / 0	8 / 0	1 / 100	100
	100	60	80	120	

The cost is 1940.

Vogel's method with Larson's modification

- An improvement of Vogel's method can be achieved if we normalize the cost matrix according to $c'_{ij} = c_{ij} - \frac{1}{n}\sum_{p=1}^{n} c_{pj} - \frac{1}{m}\sum_{q=1}^{m} c_{iq}$, i.e. c'_{ij} is obtained from c_{ij} by subtracting the average of the costs of the row and the column in which it appears.
- The use Vogel's method with the differences calculated using c'_{ij} instead of c_{ij}.