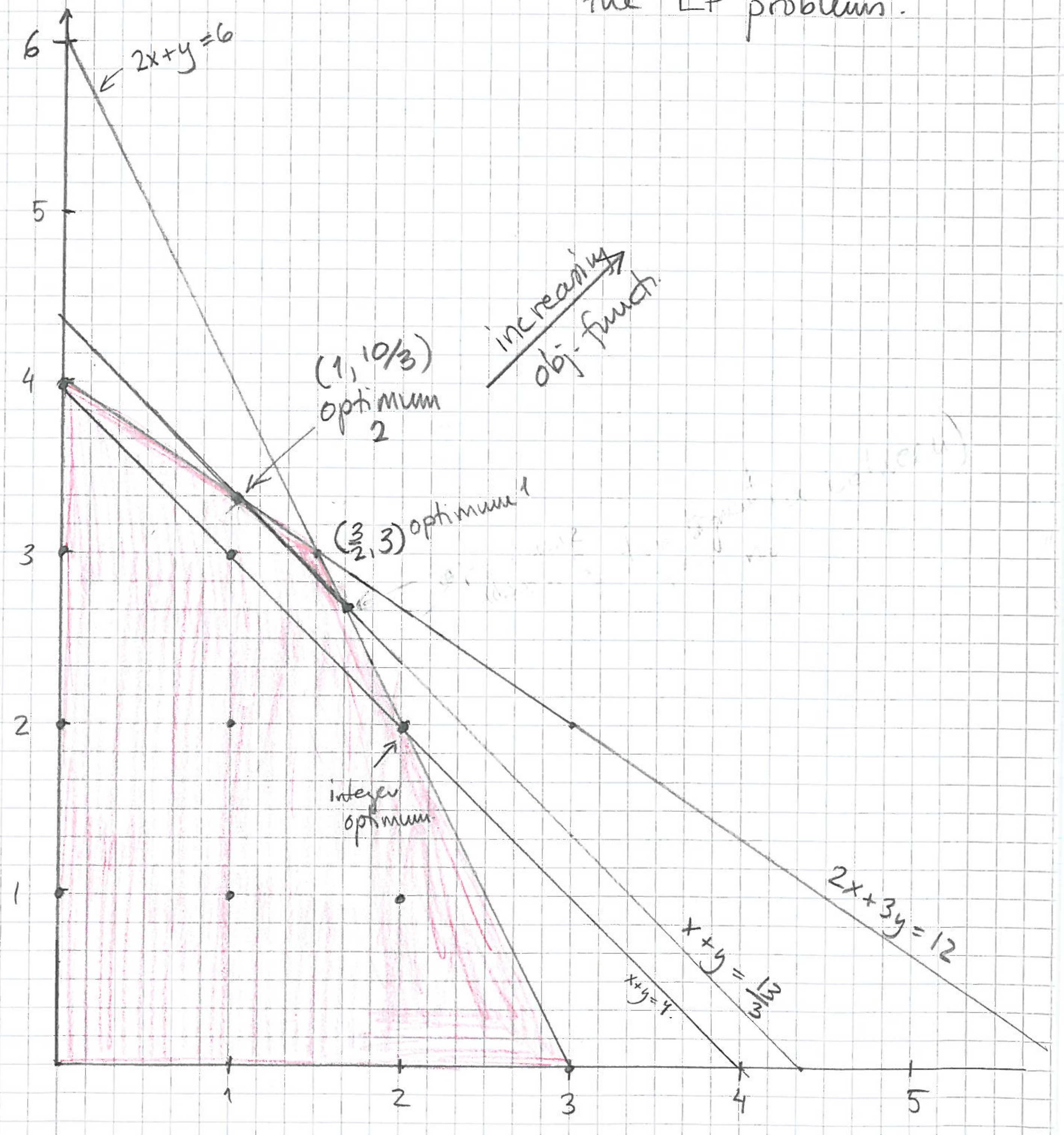
4.2.3

$$\max z = x + y$$
$$\text{s.t.} \begin{cases} 2x + 3y \le 12 \\ 2x + y \le 6. \end{cases}$$

Alt. 1. Solution with cutting planes using a graphical method for the LP problem.



The max pt $(3, 3/2)$ was found graphically.
Let's construct the first cutting plane.
We need to find the tableau for the soln $(\frac{3}{2}, 3)$.
With slack vars we have:

$$\begin{cases} 2x + 3y + u = 12 \\ 2x + y + v = 6 \end{cases} \iff \begin{cases} 2y + u - v = 6 \\ 4x - u + 3v = 6 \end{cases} \iff$$

$$\begin{cases} y + \frac{1}{2}u - \frac{1}{2}v = 3 \\ x - \frac{1}{4}u + \frac{3}{4}v = \frac{3}{2}. \end{cases}$$ Nu är x & y basvariabler & $x = \frac{3}{2}$, $y = 3$

Need to choose the 2nd eqn to form the cutting plane. (or line in this case).

$$x - \frac{1}{4}u + \frac{3}{4}v = \frac{3}{2} \iff \underbrace{x - u}_{A} + \underbrace{\frac{3}{4}u + \frac{3}{4}v}_{B} = 1 + \frac{1}{2}$$

We get the new ineq: $\frac{3}{4}u + \frac{3}{4}v \geq \frac{1}{2}$

$u = 12 - 2x - 3y$ & $v = 6 - 2x - y$ gives an inequality for x, y:

$$\frac{3}{4}(12 - 2x - 3y) + \frac{3}{4}(6 - 2x - y) \geq \frac{1}{2} \iff$$

$$9 - \frac{3}{2}x - \frac{9}{4}y + \frac{9}{2} - \frac{3}{2}x - \frac{3}{4}y \geq \frac{1}{2} \iff$$

$$-3x - 3y \geq -13 \iff 3x + 3y \leq 13$$

NB! Don't simplify to $x + y \leq \frac{13}{3}$!

Geometrically, we can solve the new system of inequalities with this new ineq included.
The maximum occurs on the segment between $(\frac{5}{3}, \frac{8}{3})$ and $(1, \frac{10}{3})$. Pick one of these extreme points. E.g. $(1, \frac{10}{3})$

$$\begin{cases} x - \frac{1}{2}u + \frac{3}{4}v = \frac{3}{2} \\ y + \frac{1}{2}u - \frac{1}{2}v = 3 \\ 3x + 3y + w = 13 \end{cases}$$

The basic feasible soln. corresponds to $u = 0$ & $w = 0$.

New basic vars are x, y & v. Solve for those variables: (Gauss elimination)

$$\begin{cases} x \qquad -\frac{1}{2}u + \frac{3}{4}v = \frac{3}{2} \\ \quad y + \frac{1}{2}u - \frac{1}{2}v = 3 \\ \qquad\qquad -\frac{3}{4}v + w = -\frac{1}{2} \end{cases} \iff$$

$$\begin{cases} x \qquad -\frac{1}{2}u \qquad + w = 1 \\ \quad y + \frac{1}{2}u \qquad -\frac{2}{3}w = \frac{10}{3} \\ \qquad\qquad v - \frac{4}{3}w = \frac{2}{3} \end{cases}$$

$\leftarrow$ Use this (for example) to get a cutting plane

$$v - \frac{4}{3}w = \frac{2}{3} \iff$$

$$\underbrace{v - 2w}_{A} + \underbrace{\frac{2}{3}w}_{B} = 0 + \frac{2}{3}.$$

$$A \in \mathbb{Z} \quad B \geq 0 \implies \frac{2}{3}w \geq \frac{2}{3}$$ new cutting plane.

Using $w = 13 - 3x - 3y$, we get

$$13 - 3x - 3y \geq 1 \iff 3x + 3y \leq 12 \iff x + y \leq 4.$$

Grafically, we see that the optimum is $\begin{cases} x = 2 \\ y = 2 \end{cases}$ & $z = 4$. Which is an integer solution.

4.2.3 **Alt 2** Solution using simplex + dual system.

1st tableau:

$$\begin{array}{cccc|c} 2 & 3 & 1 & 0 & 12 \\ 2 & 1 & 0 & 1 & 6 \\ \hline -1 & -1 & 0 & 0 & 0 \end{array} \sim \begin{array}{cccc|c} 0 & 2 & 1 & -1 & 6 \\ 1 & 1/2 & 0 & 1/2 & 3 \\ \hline 0 & -1/2 & 0 & 1/2 & 3 \end{array} \sim$$

$$\begin{array}{cccc|c} 0 & 1 & 1/2 & -1/2 & 3 \\ 1 & 0 & -1/4 & 3/4 & 3/2 \\ \hline 0 & 0 & 1/4 & 1/4 & 9/2 \end{array} \longleftarrow \underbrace{x_1 - x_3}_{A\in\mathbb{Z}} + \underbrace{\frac{3}{4}x_3 + \frac{3}{4}x_4}_{B\geq 0} = 1 + \frac{1}{2}$$

nonintger opt. Gives cutting plane ineq. $\frac{3}{4}x_3 + \frac{3}{4}x_4 \geq \frac{1}{2}$.

$\rightarrow \frac{3}{4}x_3 + \frac{3}{4}x_4 - x_5 = \frac{1}{2}$

new slack var $\in \mathbb{Z}$.

New tableau:

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	1/2	−1/2	0	3
x_1	1	0	−1/4	3/4	0	3/2
x_5	0	0	−3/4	−3/4	1	−1/2
	0	0	1/4	1/4	0	9/2

infeasible point! Use the dual!

Basic variables for dual system: y_3 & y_4. The rest are nonbasic. Dual tableau.

	y_1	y_2	y_3	y_4	y_5	
y_3	1/4	−1/2	1	0	3/4	1/4
y_4	−3/4	1/2	0	1	3/4	1/4
	3/2	3	0	0	−1/2	−9/2

Can use either y_3 or y_4 as outgoing vars. $\sim$ Take y_3 (Bland's rule)

	y_1	y_2	y_3	y_4	y_5	
y_5	1/3	−2/3	4/3	0	1	1/3
y_4	−1	1	−1	1	0	0
	5/3	8/3	2/3	0	0	−13/3

Back to primal tableau

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	1	−1/3	5/3 ←
x_2	0	1	0	−1	2/3	8/3
x_3	0	0	1	1	−4/3	2/3
	0	0	0	0	1/3	13/3

Construct a cutting plane with the help of the 1st equation for example

$$\underbrace{x_1 + x_4 - x_5}_{A} + \underbrace{\frac{2}{3}x_5}_{B} = 1 + \frac{2}{3}$$

gives the cutting plane ineq. $\frac{2}{3}x_5 \geq \frac{2}{3}$.

new slack var.

$$\frac{2}{3}x_5 \geq \frac{2}{3} \Rightarrow \frac{2}{3}x_5 - x_6 = \frac{2}{3} \Leftrightarrow -\frac{2}{3}x_5 + x_6 = -\frac{2}{3}.$$

New system.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	1	$-1/3$	0	5/3
x_2	0	1	0	-1	2/3	0	8/3
x_3	0	0	1	1	$-4/3$	0	2/3
x_6	0	0	0	0	$-2/3$	1	$-2/3$
z	0	0	0	0	1/3	0	13/3

Go to the dual!

	y_1	y_2	y_3	y_4	y_5	y_6	
y_4	-1	1	-1	1	0	0	0
y_5	1/3	$-2/3$	4/3	0	1	2/3	1/3
	5/3	8/3	2/3	0	0	$-2/3$	$-13/3$

$\sim$

	y_1	y_2	y_3	y_4	y_5	y_6	
y_4	-1	1	-1	1	0	0	0
y_6	1/2	-1	2	0	3/2	1	1/2
	2	2	2	0	1	0	-4

optimal (all integer)

Back to primal:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	1	0	$-1/2$	2
x_2	0	1	0	-1	0	1	2
x_3	0	0	1	1	0	-2	2
x_5	0	0	0	-1	1	0	1
z	0	0	0	0	0	1/2	4

$\begin{cases} x_1 = 2 \\ x_2 = 2 \end{cases}$ in original variables

gives $\underline{\underline{z = 4}}$.

4.3.1 Solve with branch & bound

$\max\ z = x + y$

subj. to $\begin{cases} 2x_1 + 3x_2 \le 12 \\ 2x_1 + x_2 \le 6 \\ x_1, x_2 \ge 0 \text{ integers} \end{cases}$

Solve ~~with~~ simplex. We have already done this in exercise 4.2.3.

Optimal solution: $(x_1, x_2) = (\frac{3}{2}, 3)$.

Use x_1 as a branch variable. $x_1 \le 1$ or $x_1 \ge 2$.

$x_1 \le 1$: $\underbrace{x_1 = \frac{3}{2} + \frac{1}{4}x_3 - \frac{3}{4}x_4}_{\text{from last tableau.}} \longrightarrow \frac{1}{4}x_3 - \frac{3}{4}x_4 \le -\frac{1}{2}$

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	1/2	−1/2	0	3
x_1	1	0	−1/4	3/4	0	3/2
x_5	0	0	1/4	−3/4	1	−1/2
~~z~~	0	0	1/4	1/4	0	9/2

Dual problem:

	y_1	y_2	y_3	y_4	y_5	
y_3	1/4	−1/2	1	0	−1/4	1/4
y_4	−3/4	1/2	0	1	3/4	1/4
	3/2	3	0	0	−1/2	−9/2

~

	y_1	y_2	y_3	y_4	y_5	
y_3	0	−1/3	1	1/3	0	1/3
y_5	−1	2/3	0	4/3	1	1/3
	1	10/3	0	2/3	0	−13/3

optimal

Primal tableau:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	0	1	1
x_2	0	1	1/3	0	−2/3	10/3
x_4	0	0	−1/3	1	−4/3	2/3
z	0	0	1/3	0	1/3	13/3

$x_1 \ge 2$

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	1/2	−1/2	0	3
x_1	1	0	−1/4	3/4	0	3/2
x_5	0	0	−1/4	3/4	1	−1/2
z	0	0	1/4	1/4	0	9/2

Dual:

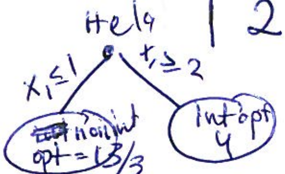
	y_1	y_2	y_3	y_4	y_5	
y_3	1/4	−1/2	1	0	1/4	1/4
y_4	−3/4	1/2	0	1	−3/4	1/4
	3/2	3	0	0	−1/2	−9/2

	y_1	y_2	y_3	y_4	y_5	
y_5	1	−2	4	0	1	1
y_4	0	−1	3	1	0	1
	2	2	2	0	0	−4

integer opt.

Primal

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	0	−1	2
x_2	0	1	0	1	2	2
x_3	0	0	1	−3	−4	2
z	0	0	0	1	1	4



$= 4 + \frac{1}{3} \Rightarrow$ Övre gräns = 4.

We cannot find a better optimum in the left branch. Opt. value is 4, $x_1 = 2, x_2 = 2$