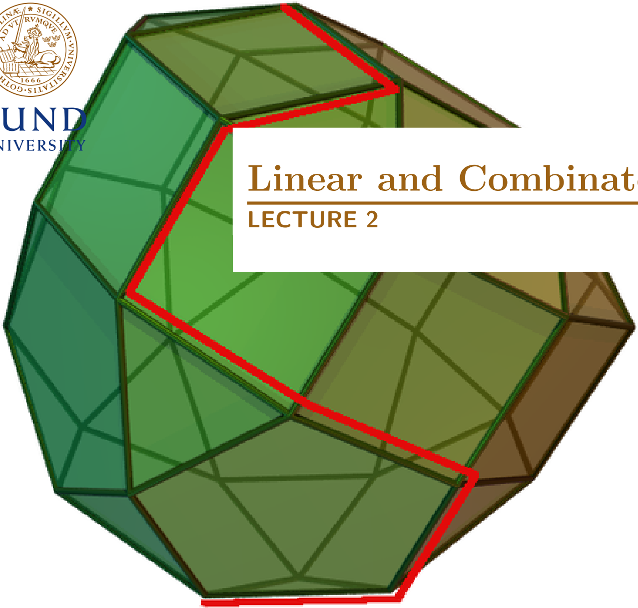
LUND UNIVERSITY



Linear and Combinatorial Optimization 2020

LECTURE 2

Overview

LUND UNIVERSITY

Convex optimization

Last time, we discussed convex sets and convex optimization problems:

Definition

A problem of the form

$$\begin{cases} \textit{minimize} & z = f(\mathbf{x}), \\ \textit{subject to} & \mathbf{x} \in C, \end{cases}$$

where $C \subset \mathbb{R}^n$ *is convex, closed and* $f : C \to \mathbb{R}$ *is convex, is called a* convex optimization problem.

Note that convex optimization problems are always minimization problems!

Linear programming

We also defined the general linear programming problem:

The general problem of linear programming.

$$\text{Maximize or minimize} \qquad z = \mathbf{c}^{\mathsf{T}}\mathbf{x}$$

$$\text{subject to} \quad \begin{cases} \mathbf{A_1 x} \leq \mathbf{b_1}, \\ \mathbf{A_2 x} \geq \mathbf{b_2}, \\ \mathbf{A_3 x} = \mathbf{b_3}, \end{cases}$$

where $\mathbf{c}$, $\mathbf{A_j}$, $\mathbf{b_j}$ are given vectors or matrices of size $n \times 1$, $m_j \times n$, $m_j \times 1$, respectively (but all of them doesn't have to be there). The relations $\leq$, $\geq$ and $=$ are taken componentwise. $\mathbf{x} \in \mathbb{R}^n$ is the unknown vector that we wish to find together with the optimal value z.

Linear programming

Example (Saw mill application, p. 46 in Kolman–Beck)

A saw mill produces two types of lumber: finish grade *and* construction grade. *The saw is available 8 hours per day, and the plane is available 15 hours per day.*

It takes 2 hours to rough saw 1000 board feet for both types (1 board foot = 1 foot × 1 foot × 1 inch).

It takes 5 hours to plane 1000 board feet of finish grade, but only 3 hours to plane 1000 board feet of construction grade.

The profit is $120 for finish grade and $100 for construction grade for 1000 board feet.

How much of each type should the saw mill produce in order to maximize the profit?

Linear programming

We formulate the problem of the saw mill example as an LP problem. Let x_1, x_2 be the amounts of finish grade and construction grade lumber, produced in one day.
The total saw time per day cannot be more than 8 hours gives the constraint

$$2x_1 + 2x_2 \leq 8.$$

Similarly, the total plane time per day cannot be more than 15 hours, that is

$$5x_1 + 3x_2 \leq 15.$$

We must also have $x_1, x_2 \geq 0$, since the amounts produced cannot be negative.
Finally, the profit should be maximized, and so we should maximize

$$z = 120x_1 + 100x_2.$$

Linear programming

Therefore, the saw mill problem can be stated as the LP problem

$$\text{maximize} \quad z = 120x_1 + 100x_2,$$
$$\text{subject to} \begin{cases} 2x_1 + 2x_2 \leq 8, \\ 5x_1 + 3x_2 \leq 15, \\ x_1, x_2 \geq 0. \end{cases}$$

We will now start learning how to solve this and other LP problems.

LUND UNIVERSITY

Different forms of an LP problem

Definition (LP problem on standard form)

An LP problem of the form

$$\begin{aligned} &\textit{maximize} \quad z = \mathbf{c}^{\mathsf{T}}\mathbf{x}, \\ &\textit{subject to} \begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases} \end{aligned}$$

is said to be on standard form

- As you can see, it is a maximization problem with only inequality ($\leq$) constraints, and all variables are required to be non-negative.
- Note that the LP problem of the saw mill example is on standard form.

Different forms of an LP problem

Definition (LP problem on canonical form)

An LP problem of the form

$$\begin{aligned} &\textit{maximize} \quad z = \mathbf{c}^{\mathsf{T}}\mathbf{x}, \\ &\textit{subject to} \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases} \end{aligned}$$

is said to be on canonical form

- As you can see, it is a maximization problem with only equality constraints, and all variables are required to be non-negative.

Changing the form of an LP problem

All LP problems can be converted into standard and canonical forms by using the following operations:

Changing min to max.

$$\min z = \mathbf{c}^\mathsf{T}\mathbf{x} \quad \iff \quad \max(-z) = -\mathbf{c}^\mathsf{T}\mathbf{x}.$$

Convert $\geq$ to $\leq$.

Multiply the $\geq$ constraint(s) by -1:

$$\mathbf{A}\mathbf{x} \geq \mathbf{b} \quad \iff \quad -\mathbf{A}\mathbf{x} \leq -\mathbf{b}.$$

LUND UNIVERSITY

Changing the form of an LP problem

Convert $=$ to $\leq$.

$$\mathbf{Ax} = \mathbf{b} \quad \iff \quad \mathbf{Ax} \leq \mathbf{b} \text{ and } \mathbf{Ax} \geq \mathbf{b} \quad \iff$$

$$\begin{cases} \mathbf{Ax} \leq \mathbf{b} \\ -\mathbf{Ax} \leq -\mathbf{b} \end{cases} \iff \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$$

This replaces the equality constraints with twice as many inequality constraints.

Changing the form of an LP problem

Convert $\leq$ to $=$.

The system of constraints

$$\begin{cases} \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

can be converted to a system of constraints of the form

$$\begin{cases} \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \mathbf{b} \\ \tilde{\mathbf{x}} \geq \mathbf{0} \end{cases}$$

as follows: Let

$$\mathbf{u} = \mathbf{b} - \mathbf{A}\mathbf{x},$$

and note that $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ if and only if $\mathbf{u} \geq 0$. The entries of $\mathbf{u}$ are called slack variables.

Changing the form of an LP problem

Convert $\leq$ to $=$ (Cont.)

Hence

$$\begin{cases} \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \iff \begin{cases} \mathbf{Ax} + \mathbf{u} = \mathbf{b} \\ \mathbf{x}, \mathbf{u} \geq \mathbf{0} \end{cases}$$

in the sense that if the system on the left holds for some $\mathbf{x}$, then there exists a $\mathbf{u}$ such that the right system holds for $\mathbf{x}$, $\mathbf{u}$. The system to the right above is equivalent to

$$\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{b}, \qquad \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \geq \mathbf{0}.$$

Let

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}$$

and we are done.

Changing the form of an LP problem

Replacing unconstrained variables by constrained ones.

We would like to convert $\mathbf{x} \in \mathbb{R}^m$ to $\tilde{\mathbf{x}} \geq 0$ for some vector $\tilde{\mathbf{x}}$ of suitable length. Write $\mathbf{x} = \mathbf{x}_+ - \mathbf{x}_-$, where $\mathbf{x}_\pm \geq \mathbf{0}$ (and $\mathbf{x}_\pm \in \mathbb{R}^m$). This is always possible, but $\mathbf{x}_+$ and $\mathbf{x}_-$ are not unique. We have

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \iff \mathbf{A}(\mathbf{x}_+ - \mathbf{x}_-) \leq \mathbf{b} \iff$$
$$\mathbf{A}\mathbf{x}_+ - \mathbf{A}\mathbf{x}_- \leq \mathbf{b} \iff \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}_+ \\ \mathbf{x}_- \end{bmatrix} \leq \mathbf{b}.$$

Putting $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix}$ and $\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_+ \\ \mathbf{x}_- \end{bmatrix}$, we see that we have succeeded if we at the same time replace the constraint $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ by $\tilde{\mathbf{A}}\tilde{\mathbf{x}} \leq \mathbf{b}$.

Saw mill example, revisited

Linear programming

The Simplex algorithm

Example

We have formulated the saw mill example as an LP problem in standard form. Let us convert it to canonical form. The original standard form problem is

$$\begin{aligned} &\textit{maximize} \quad z = 120x_1 + 100x_2, \\ &\textit{subject to} \begin{cases} 2x_1 + 2x_2 \leq 8, \\ 5x_1 + 3x_2 \leq 15, \\ \quad x_1, x_2 \geq 0. \end{cases}. \end{aligned}$$

Following the recipe for converting an inequality to an equality, we introduce slack variables u_1 and u_2 by

$$\begin{cases} u_1 = 8 - 2x_1 - 2x_2, \\ u_2 = 15 - 5x_1 - 3x_2. \end{cases}$$

Saw mill example, revisited (Cont.)

Example

The original system is then equivalent to

$$\begin{aligned} \textit{maximize} \quad & z = 120x_1 + 100x_2, \\ \textit{subject to} \quad & \begin{cases} 2x_1 + 2x_2 + u_1 = 8, \\ 5x_1 + 3x_2 + u_2 = 15, \\ x_1, x_2, u_1, u_2 \geq 0, \end{cases} . \end{aligned}$$

which is on canonical form.

The Simplex method

Consider a problem in canonical form

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^\mathsf{T}\mathbf{x}, \\ \text{subject to} \quad & \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases}, \end{aligned}$$

where $\mathbf{A}$ is an $m \times n$-matrix, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$.
We need to make the basic assumptions that

- $m \leq n$,
- A has rank m, i.e. A has m linearly independent columns.

The basic assumptions always hold if the problem has been converted from standard form.

Basic and nonbasic variables

We start by identifying m columns of $\mathbf{A}$ which are linearly independent. We assume that we have numbered the variables so that they are the last m columns of $\mathbf{A}$, and denote this submatrix by $\mathbf{A}_B$ (B as in *basic*), and the submatrix which is formed by the remaining columns by $\mathbf{A}_N$ (N as in non-basic).

Similarly, we write $\mathbf{x} = \begin{bmatrix} \mathbf{x}_N \\ \mathbf{x}_B \end{bmatrix}$.

Definition

- *If* $\mathbf{x}$ *satisfies the constraint* $A\mathbf{x} = \mathbf{b}$*, then we say that* $\mathbf{x}$ *is a* solution*.*
- *If* $\mathbf{x}$ *is a solution for which* $\mathbf{x}_N = \mathbf{0}$*, then* $\mathbf{x}$ *is said to be a* basic solution*.*
- *If* $\mathbf{x}$ *is a basic solution for which* $\mathbf{x}_B \geq \mathbf{0}$*, then* $\mathbf{x}$ *is said to be a* basic feasible solution*.*

Basic feasible solutions

- We note that if $\mathbf{x}$ is a basic feasible solution, then $\mathbf{x} \geq \mathbf{0}$, since $\mathbf{x}_B \geq \mathbf{0}$ and $\mathbf{x}_N = 0$. Since a basic feasible solution is a solution, $\mathbf{x}$ also satisfies $A\mathbf{x} = \mathbf{b}$, and so satisfies all the constraints of the maximization problem.
- A basic solution can always be written as $\mathbf{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_B^{-1}\mathbf{b} \end{bmatrix}$. Indeed, if $\mathbf{x} = \begin{bmatrix} \mathbf{x}_N \\ \mathbf{x}_B \end{bmatrix}$ is a basic solution, then $\mathbf{x}_N = \mathbf{0}$ and so

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \Longleftrightarrow \quad \begin{bmatrix} \mathbf{A}_N & \mathbf{A}_B \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{x}_B \end{bmatrix} = \mathbf{b} \quad \Longleftrightarrow$$

$$\mathbf{A}_N\mathbf{0} + \mathbf{A}_B\mathbf{x}_B = \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{A}_B\mathbf{x}_B = \mathbf{b}.$$

Solving for $\mathbf{x}_B$ and recalling that $\mathbf{x}_N = 0$, we obtain the required expression for $\mathbf{x}$.

Basic feasible solutions and extreme points

Theorem

A basic feasible solution is an extreme point of the convex polyhedron that describes the feasible set of the LP problem.

- The above theorem connects the geometric property of extreme points of a convex polyhedron to the algebraic property of basic feasible solutions.
- The extreme point theorem for LP problems says that if an optimal solution of an LP problem exists, then there has to be one at an extreme point, and hence at a basic feasible solution.
- The Simplex method takes advantage of this. Starting at a basic feasible solution (=an extreme point), we move along an edge of the convex polyhedron that describes the feasible set to an adjecent extreme point of the polyhedron, making sure that the objective value increases in each step, and iterating until an optimal solution has been found.

The Simplex method

Example (The simplex method for the saw mill example)

Let's recall the statement of the problem in canonical form:

$$\begin{aligned} &\textit{maximize} \quad z = 120x_1 + 100x_2, \\ &\textit{subject to} \begin{cases} 2x_1 + 2x_2 + u_1 = 8, \\ 5x_1 + 3x_2 + u_2 = 15, \\ x_1, x_2, u_1, u_2 \geq 0, \end{cases} . \end{aligned}$$

We take u_1 and u_2 as basic variables and x_1 and x_2 as nonbasic variables to start with. If we put the nonbasic variables are set to 0, we immediately see that $u_1 = 8$ and $u_2 = 15$, which are both nonnegative, and so $(x_1, x_2, u_1, u_2)^T = (0, 0, 8, 15)^T$ is our first basic feasible solution. Let us organize our calculations in a table as follows:

The Simplex method

Example (The simplex method for the saw mill example, Cont.)

$$
\begin{array}{c|ccccccccc}
\hline
u_1 & 2x_1 & + & 2x_2 & + & u_1 & & & = & 8 \\
u_2 & 5x_1 & + & 3x_2 & & & + & u_2 & = & 15 \\
\hline
z & -120x_1 & - & 100x_2 & & & & + z & = & 0
\end{array}
$$

The rows are labelled (in the leftmost column) by the basic variable that is solved for in that row. The last row is the objective row. We can see that the value for z is 0 for this solution if we put the nonbasic variables to 0. Now we would like to change to another basic feasible solution in such a way that the objective value becomes higher. In order to do that, we look in the objective row, and see if one of the nonbasic variables can be increased (and turned into a basic variable) while increasing the value of z.

The Simplex method

Example (The simplex method for the saw mill example, Cont.)

Indeed, if e g x_1 *is increased, and* x_2 *is kept at* 0*, then* z *has to increase in order for the equation in the objective row to be satisfied. We call* x_1 *our incoming variable, since it will be a new basic variable. (We could instead have chosen* x_2 *to be our incoming variable by increasing* x_2 *and keeping* $x_1 = 0$*). We next need to find out how much we can increase* x_1*. We can increase* x_1 *until* u_1 *or* u_2 *becomes* 0*. The question is which one of these variables will become* 0 *first.*
In this case we see from the first two rows (after putting $x_2 = 0$*) that*

$$\begin{cases} 2x_1 + u_1 = 8 \\ 5x_1 + u_2 = 15 \end{cases} \iff \begin{cases} u_1 = 8 - 2x_1 \\ u_2 = 15 - 5x_1. \end{cases}$$

Example (The simplex method for the saw mill example, Cont.)

Hence, u_1 and u_2 are both non-negative if and only if $8 - 2x_1 \geq 0$ and $15 - 5x_1 \geq 0$, i e if and only if $x_1 \leq 4$ and $x_1 \leq 3$. This shows that u_2 becomes 0 first (when $x_1 = 3$), and so u_2 will be a new non-basic variable. It is called an outgoing variable. Next, we solve for the incoming variable x_1 in the row of the outgoing variable u_2:

$$x_1 = \tfrac{1}{5}(15 - 3x_2 - u_2) = 3 - \tfrac{3}{5}x_2 - \tfrac{1}{5}u_2.$$

Next, we substitute this expression of the RHS for x_1 into the other rows:

$$\begin{aligned} 2\left(3 - \tfrac{3}{5}x_2 - \tfrac{1}{5}u_2\right) + 2x_2 + u_1 = 8 \quad &\Leftrightarrow \quad \tfrac{4}{5}x_2 + u_1 - \tfrac{2}{5}u_2 = 2 \\ -120\left(3 - \tfrac{3}{5}x_2 - \tfrac{1}{5}u_2\right) - 100x_2 + z = 0 \quad &\Leftrightarrow \quad -28x_2 + 24u_2 + z = 360. \end{aligned}$$

Arranging the final equations in a new table, we obtain:

Example (The simplex method for the saw mill example, Cont.)

u_1			$\frac{4}{5}x_2$	$+$	u_1	$-$	$\frac{2}{5}u_2$			$=$	2
x_1	x_1	$+$	$\frac{3}{5}x_2$			$+$	$\frac{1}{5}u_2$			$=$	3
z		$-$	$28x_2$			$+$	$24u_2$	$+$	z	$=$	360

We see that the value has increased from 0 to 360. This was one step in the Simplex algorithm.

In the next step, we need to choose a new incoming variable, and this has to be x_2 since if x_2 is increasing, then z will also increase, while if we increase u_2, then z will decrease.

Performing the same steps as above, we see that we can increase x_2 as long as

$$\begin{cases} u_1 = 2 - \frac{4}{5}x_2 \geq 0 \\ x_1 = 3 - \frac{3}{5}x_2 \geq 0. \end{cases} \iff \begin{cases} x_2 \leq \frac{5}{2} \\ x_2 \leq 5. \end{cases}$$

LUND UNIVERSITY

Example (The simplex method for the saw mill example, Cont.)

Hence, the variable that will become 0 *first is* u_1, *and so* u_1 *is the new outgoing variable.*

The new table is

$$
\begin{array}{c|cccccccc}
\hline
x_2 & & x_2 & + & \frac{5}{4}u_1 & - & \frac{1}{2}u_2 & & = & \frac{5}{2} \\
x_1 & x_1 & & - & \frac{3}{4}u_1 & + & \frac{1}{2}u_2 & & = & \frac{3}{2} \\
\hline
z & & & & 35u_1 & + & 10u_2 & + \; z & = & 430
\end{array}
$$

Now it is not possible to choose an incoming variable. An optimal solution has been found. (We will prove this next time). The optimal value is 430, *and we have found an optimal solution* $(x, y) = \left(\frac{3}{2}, \frac{5}{2}\right)$.

The Simplex algorithm

We have seen how to solve an LP problem in canonical form for the example of the saw mill problem, which was of the form

$$\begin{aligned} &\text{maximize} \quad z = \mathbf{c}^T \mathbf{x}, \\ &\text{subject to} \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b}, \\ \quad \mathbf{x} \geq \mathbf{0}, \end{cases} \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{n \times m}$ $(n \geq m)$, and where we assume that $\mathbf{A}$ has rank m. The method that were used always works if we somehow are able to find a first basic feasible solution. This is easiest when $\mathbf{b} \geq \mathbf{0}$, so we will assume this at first, and handle the general case next time.

The Simplex method

As before, we write $\mathbf{A} = [\mathbf{A}_N, \mathbf{A}_B]$, and recall that

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{A}_N\mathbf{x}_N + \mathbf{A}_B\mathbf{x}_B = \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x}_B = \mathbf{A}_B^{-1}(b - \mathbf{A}_N\mathbf{x}_N).$$

If $\mathbf{x}_N = 0$ then $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b}$ and we have a *basic solution*. If in addition, $\mathbf{x}_B \geq \mathbf{0}$, then we have a *basic feasible solution*.
We express the objective function in terms of only the non-basic variables:

$$\begin{aligned} z = \mathbf{c}^T\mathbf{x} = \mathbf{c}_N^T\mathbf{x}_N + \mathbf{c}_B^T\mathbf{x}_B &= \mathbf{c}_N^T\mathbf{x}_N + \mathbf{c}_B^T\mathbf{A}_B^{-1}(b - \mathbf{A}_N\mathbf{x}_N) \\ &= (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_N)\mathbf{x}_N + \mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}. \end{aligned}$$

Linear programming

We organize the computations in a table as we did in the example, but now also using matrices:

$\mathbf{x}_B$	$\mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N$	$+\ \mathbf{x}_B$		$=$	$\mathbf{A}_B^{-1}\mathbf{b}$
z	$(\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_N - \mathbf{c}_N^T)\mathbf{x}_N$		$+\ z$	$=$	$\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$

Tableaus in Matlab

In Matlab, we organize the computations in tableau form, which is the same as the above system, except that we don't write out the variables $\mathbf{x}_B$, $\mathbf{x}_N$ and z:

$\mathbf{A}_B^{-1}\mathbf{A}_N$	$\mathbf{I}$	0	$\mathbf{A}_B^{-1}\mathbf{b}$
$\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{A}_N - \mathbf{c}_N^T$	$\mathbf{0}$	1	$\mathbf{c}_B^T\mathbf{A}_B^{-1}\mathbf{b}$

Note that in Matlab, the order of the columns will sometimes be different than the above. For example it is not always the case that the basic variables will be at the end.

The above tableau may be useful for the completion of handin exercise 1.