LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Partial Differential Equations
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Solutions

1. Separation of variables gives the solution

$$u(x,t) = \sin(2x)\cos(2t) + \frac{3}{4}\sin(x)\sin(t) - \frac{1}{12}\sin(3x)\sin(3t)$$

(note that $\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$).

2. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_U u^2\,\mathrm{d}x &= 2\int_U uu_t\,\mathrm{d}x \\
&= 2\int_U u\Delta u\,\mathrm{d}x - 2\int_U \sum_{i=1}^n b_i uu_{x_i} - 2\int_U cu^2\,\mathrm{d}x \\
&= -2\int_U |Du|^2\,\mathrm{d}x - 2\int_U cu^2\,\mathrm{d}x + 2\int_{\partial U} u\frac{\partial u}{\partial \nu}\,\mathrm{d}S - \int_{\partial U} u^2 b\cdot\nu\,\mathrm{d}S \\
&= -2\int_U |Du|^2\,\mathrm{d}x - 2\int_U cu^2\,\mathrm{d}x \\
&\le 0,
\end{aligned}$$

where $b = (b_1,\dots,b_n)$ and we have used the equalities $\operatorname{div}(u^2b) = \operatorname{div}(u^2b_1,\dots,u^2b_n) = \sum_{i=1}^n 2uu_{x_i}b_i$ and $u|_{\partial U} = 0$.

3. **a)** This is a first-order equation, which can be solved using the method of characteristics. The projected characteristics are of the form $x = ae^s$, $y = be^{-s}$ and $z = ce^{-s}$. Note that the projected characteristics don't cross the x-axis. It is convenient to choose $b = 1$ so that the characteristics cross the line $\{y = 1\}$ when $s = 0$. This implies that $y = e^{-s}$. Using the initial condition we get that $c = \sin(a)$, where $a = xe^{-s} = xy$. Hence, $c = \sin(xy)$. Finally $z = ce^{-s} = y\sin(xy)$. A direct calculation shows that this is indeed a solution to the problem. Note that the solution is uniquely determined in the upper half-plane, since every point there lies on a unique characteristic which passes through the line $\{y = 1\}$

b) It is clear that the formula $u(x,y) = y\sin(xy)$ in fact defines a solution in the whole of $\mathbb{R}^2$. So does e.g.

$$u(x,y) = \begin{cases} y\sin(xy), & y \ge 0, \\ 0, & y < 0 \end{cases}$$

(there are infinitely many different extensions). This is consistent with the fact that the characteristics don't cross the x-axis.

Please, turn over!

4. a) In order to show that L is uniformly elliptic, we need to find a constant $\theta > 0$ such that

$$\xi_1^2 - 2\xi_1\xi_2 + 2\xi_2^2 \geq \theta(\xi_1^2 + \xi_2^2)$$

for each $(\xi_1, \xi_2) \in \mathbb{R}^2$ and $(x_1, x_2) \in U$. This follows e.g. by noting that

$$\xi_1^2 - 2\xi_1\xi_2 + 2\xi_2^2 \geq \left(1 - \frac{1}{a}\right)\xi_1^2 + (2-a)\xi_2^2,$$

for any $a > 0$ and choosing $1 < a < 2$ (with $\theta = \min\{1 - \frac{1}{a}, 2-a\}$).

b) Suppose now that $u \in C^2(U) \cap C(\overline{U})$ is a solution to the problem and that $u(x) > 1$ somewhere in U. Let $V = \{x \in U : u(x) > 1\} \neq \emptyset$. Then $Lu < 0$ on V and $u = 1$ on ∂V (note that $\partial V \cap \partial U = \emptyset$). Hence, the maximum of u is $\overline{V}$ is attained on ∂V by the weak maximum principle, giving the contradiction $u \leq 1$ in V. A similar argument shows that $u \geq -1$. Thus $|u| \leq 1$ in U.

5. a) Since $u \in H^1_{\text{per}}$ it has a Fourier series expansion $u = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}$ with convergence in L^2_{per} (in fact, uniform convergence). Moreover, $u' = \sum_{k\in\mathbb{Z}} ik\hat{u}_k e^{ikx}$ with convergence in L^2_{per}. By Parseval's formula, we have that

$$\int_{-\pi}^{\pi} (u(x))^2 \, \mathrm{d}x = 2\pi \sum_{k\in\mathbb{Z}} |\hat{u}_k|^2$$

and

$$\int_{-\pi}^{\pi} (u'(x))^2 \, \mathrm{d}x = 2\pi \sum_{k\in\mathbb{Z}} k^2 |\hat{u}_k|^2.$$

Moreover, $\hat{u}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\, \mathrm{d}x = 0$ by assumption. It follows that

$$\int_{-\pi}^{\pi} (u(x))^2 \, \mathrm{d}x = 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} |\hat{u}_k|^2 \leq 2\pi \sum_{k\in\mathbb{Z}} k^2 |\hat{u}_k|^2 = \int_{-\pi}^{\pi} (u'(x))^2 \, \mathrm{d}x.$$

b) Multiply the equation by v, integrate by parts and use the boundary conditions.

c) Taking $v(x) \equiv 1$ shows the necessity. Let H be the closed subspace $\{u \in H^1_{\text{per}}(\mathbb{R}) : \int_{-\pi}^{\pi} u(x)\, \mathrm{d}x = 0\}$ of $H^1_{\text{per}}(\mathbb{R})$ (which is also a Hilbert space with the inherited inner product). Then

$$B[u,v] = \int_{-\pi}^{\pi} a(x)u'(x)v'(x)\, \mathrm{d}x, \quad u, v \in H,$$

satisfies the conditions of the Lax-Milgram theorem due to part **a**) and the conditions on $a(x)$, and

$$v \mapsto \int_{-\pi}^{\pi} f(x)v(x)\, \mathrm{d}x$$

defines a bounded linear functional on H. Hence there is a unique $u \in H$ such that

$$\int_{-\pi}^{\pi} a(x)u'(x)v'(x)\, \mathrm{d}x = \int_{-\pi}^{\pi} f(x)v(x)\, \mathrm{d}x, \tag{1}$$

for all $v \in H$ by the Lax-Milgram theorem (or in fact by Riesz' representation theorem). We have still not verified that u is a weak solution, since this requires the identity (1) to hold for all $v \in H^1_{\text{per}}$ (not just the subspace H). However, the condition $\int_{-\pi}^{\pi} f(x)\, \mathrm{d}x = 0$ guarantees that (1) continues to hold if v does not have zero average.