PDE Course 2020
Summary

May 26

Elliptic equations

$$-\Delta u = f$$

or

$$-\sum_{i,j} a^{ij}(x)\partial_{x_i}\partial_{x_j} u + \sum_i b^i(x)\partial_{x_i} u + c(x)u = f$$

or

$$-\sum_{i,j} \partial_{x_i}(a^{ij}(x)\partial_{x_j} u) + \sum_i b^i(x)\partial_{x_i} u + c(x)u = f$$

in U (often bounded domain)

Boundary conditions:

$u = g$ on ∂U (Dirichlet)

or

$\partial_\nu u = g$ on ∂U (Neumann)

etc.

Methods and results

- Explicit solution formulas
- Fundamental solution
- Green's functions
- Poisson integrals
- Mean value formulas
- Maximum principles
- Regularity
- Energy estimates
- Lax-Milgram

Parabolic equations

$$u_t - \Delta u = f$$

or

$$u_t - \sum_{i,j} a^{ij}(x)\partial_{x_i}\partial_{x_j} u + \sum_i b^i(x)\partial_{x_i} u + c(x)u = f$$

or

$$u_t - \sum_{i,j} \partial_{x_i}(a^{ij}(x)\partial_{x_j} u) + \sum_i b^i(x)\partial_{x_i} u + c(x)u = f$$

in $U_T = U \times (0, T]$

Initial-boundary condition:

$u = g$ on $\Gamma_T = \bar{U}_T - U_T = (\bar{U} \times \{0\}) \cup (\partial U \times (0, T))$

or initial condition:

$u = g$ on $U \times \{0\}$ if $U = \mathbb{R}^n$

Methods and results

- Explicit solution formulas
- Fundamental solution
- Infinite speed of propagation
- (Mean value formula)
- Maximum principles
- Regularity for $t > 0$
- Energy estimates

Hyperbolic equations

Transport: $u_t + b \cdot Du = f$

Quaslinear 1st order equations: $a(x,u) \cdot Du + b(x,u) = 0$

Conservation laws: $u_t + \operatorname{div} F(u) = 0$

Wave equation: $u_{tt} - \Delta u = f$

(or $u_{tt} - \sum_{i,j} \partial_{x_i}(a^{ij}(x)\partial_{x_j} u) + \sum_i b^i(x)\partial_{x_i} u + c(x)u = f$)

Initial conditions:

$u = g$ if 1st order

$u = g$ and $u_t = h$ if 2nd order

Methods and results

- Explicit solution formulas
- Spherical means
- Energy estimates
- Finite speed of propagation
- No gain of regularity
- Method of characteristics
- Singularities
- Weak solutions

General

- Well-posedness
- Fourier transforms
- Sobolev spaces
- Sobolev inequalities, Morrey's inquality, Poincaré's inequality
- Distributions
- Characterization of 2nd order operators (and general elliptic operators)
- Fundamental solutions and the Malgrange-Ehrenpreis theorem
- Non-characteristic surfaces
- Power series and Cauchy-Kovalevskaya

Some other books

A very selective list of other books on PDE and related topics:

General PDE

- F. John, *Partial differential equations*
- J. Rauch, *Partial differential equations*
- J. Jost, *Partial differential equations*
- M. Renardy and R. C. Rogers, *An introduction to partial differential equations*
- W. A. Strauss, Partial differential equations: an introduction
- M. Taylor, *Partial differential equations. I–III*
- G. Folland, *Introduction to partial differential equations*

Elliptic PDE

- D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*

Hyperbolic PDE

- P. D. Lax, *Hyperbolic partial differential equations*
- H. Holden and N. H. Risebro, *Front tracking for hyperbolic conservation laws*
- A. Bressan, *Hyperbolic systems of conservation laws*

Sobolev spaces

- R. A. Adams and J. J. F. Fournier, *Sobolev spaces*

Distributions and applications to PDE

- J. J. Duistermaat and J. A. C. Kolk, *Distributions*
- F. G. Friedlander, *Introduction to the theory of distributions*
- L. Hörmander, *The analysis of linear partial differential operators. I*
- L. Hörmander, *Linear partial differential operators*
- G. Grubb, *Distributions and operators*