LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Partial Differential Equations
Monday, May 28, 2018
08.00-13.00

Solutions

1. Separation of variables gives the solution

$$u(x,t) = \frac{t}{2} + \frac{1}{4}\cos(2x)\sin(2t).$$

2. By definition, we have to verify that

$$\int_{\mathbb{R}^2} u(x,t)(\partial_t\partial_x\varphi(x,t) + \partial_t\varphi(x,t))\,dx\,dt = \varphi(0,0)$$

for each $\varphi \in C_c^\infty(\mathbb{R}^2)$. We have

$$\int_{\mathbb{R}^2} u(x,t)\partial_t\varphi(x,t)\,dx\,dt = \int_0^\infty\int_0^\infty e^x\partial_t\varphi(x,t)\,dt\,dx = -\int_0^\infty e^x\varphi(x,0)\,dx.$$

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^2} u(x,t)\partial_t\partial_x\varphi(x,t)\,dx\,dt &= \int_0^\infty\int_0^\infty e^x\partial_t\partial_x\varphi(x,t)\,dt\,dx = -\int_0^\infty e^x\partial_x\varphi(x,0)\,dx\\ &= \varphi(0,0) + \int_0^\infty e^x\varphi(x,0)\,dx.\end{aligned}$$

Thus, the sum of the two terms equals $\varphi(0,0)$, as required.

3. Following the hint, we have

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,dy.$$

Differentiating under the integral sign with respect to x_j, we find that

$$\begin{aligned}\partial_{x_j}u(x,t) &= -\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}\frac{(x_j-y_j)}{2t}e^{-\frac{|x-y|^2}{4t}}g(y)\,dy\\ &= -\frac{1}{2\sqrt{t}}\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}\frac{(x_j-y_j)}{\sqrt{t}}e^{-\frac{|x-y|^2}{4t}}g(y)\,dy.\end{aligned}$$

Making the change of variables $z = (x-y)/\sqrt{t}$, we obtain

$$\partial_{x_j}u(x,t) = -\frac{1}{2\sqrt{t}}\frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^n} z_j e^{-\frac{|z|^2}{4}}g(x-\sqrt{t}z)\,dz.$$

Hence, we get

$$|\partial_{x_j}u(x,t)| \le \frac{C\|g\|_\infty}{\sqrt{t}},$$

with

$$C = \frac{1}{2(4\pi)^{n/2}}\int_{\mathbb{R}^n}|z_j|e^{-\frac{|z|^2}{4}}\,dz.$$

Please, turn over!

4. a) We use the method of characteristics. The characteristic equations are

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -1, \\ \dot{z} = z^2. \end{cases}$$

Solving the first two equations we obtain $x(s) = x^0 e^s$, $y(s) = -s + y^0$. Since we have the boundary condition $u(x,0) = g(x)$, it is convenient to choose $y^0 = 0$. We then use x^0 to parametrize the x-axis and require that $z(0) = g(x^0)$ to enforce the boundary condition $u(x,0) = g(x)$. The equation $\dot{z} = z^2$ is separable, and we find that

$$z(s;x^0) = \frac{z(0;x^0)}{1 - z(0;x^0)s} = \frac{g(x^0)}{1 - g(x^0)s}.$$

This gives a solution formula in terms of the characteristic coordinates x^0 and s. In order to express the solution in terms of x and y, we solve the equations $x(s;x^0) = x$ and $y(s;x^0) = y$ for s and x^0, finding that $s = -y$ and $x^0 = xe^{-s} = xe^y$. Thus,

$$u(x,y) = \frac{g(xe^y)}{1 + yg(xe^y)}. \tag{1}$$

Computing

$$u(x,0) = g(x),$$

$$u_x(x,y) = \frac{g'(xe^y)e^y}{1 + yg(xe^y)} - \frac{g(xe^y)g'(xe^y)ye^y}{(1 + yg(xe^y))^2}$$

and

$$u_y(x,y) = \frac{g'(xe^y)xe^y}{1 + yg(xe^y)} - \frac{g(xe^y)g'(xe^y)xye^y}{(1 + yg(xe^y))^2} - \frac{(g(xe^y))^2}{(1 + yg(xe^y))^2},$$

we see that u really is a solution. Note that the denominator $1 + yg(xe^y)$ is strictly positive for $y \geq 0$ since $g \geq 0$.

b) Since each point in $\mathbb{R}^2$ lies on a unique projected characteristic intersecting the x-axis, it follows from the method of characteristics that any solution has to be of the form (1) and that it is defined as long as $yg(xe^y) > -1$. Note that the projected characteristics can be expressed as $x = x^0 e^{-y}$. Thus $g(xe^y)$ is constant along each projected characteristic (equal to $g(x^0)$). If $g(x^0) < 0$, we therefore find that u will blow-up along the projected characteristic through $(x^0, 0)$ at $y = -\frac{1}{g(x^0)}$ (and $x = x^0 e^{\frac{1}{g(x_0)}}$). In particular, if g is bounded from below, the solution will exist on the maximal set $\mathbb{R} \times [0, y_{\max})$ where

$$y_{\max} = -\frac{1}{\inf_{x \in \mathbb{R}} g(x)}.$$

5. a) Suppose that u and v are both solutions and set $w = u - v$. We have to show that $w = 0$. Note that w solves the problem

$$Lw = 0 \quad \text{in } U,$$

$$\frac{\partial w}{\partial \nu} + w = 0 \quad \text{on } \partial U.$$

If w does not vanish identically, we must either have $\max_{\overline{U}} w > 0$ or $\min_{\overline{U}} w < 0$ (or both). Assume that $\max_{\overline{U}} w > 0$ (otherwise consider $-w$). It follows from the weak maximum principle that w attains its maximum on ∂U. Thus there is a point $x^0 \in \partial U$ with $\max_{\overline{U}} w = w(x^0) > 0$. At such a point we must also have $\frac{\partial w}{\partial \nu}(x^0) \geq 0$ since $w(x) \leq w(x^0)$ for all $x \in U$. But then

$$\frac{\partial w}{\partial \nu}(x^0) + w(x^0) > 0,$$

contradicting the boundary condition.

b) We can e.g. take $n = 1$, $U = (-1,1)$ and $Lu = -u''$ (an ordinary differential operator). Then $u(x) = x$ satisfies $Lu = 0$ for all x and $u'(1) - u(1) = 1 - 1 = 0$, $-u'(-1) - u(-1) = -1 - (-1) = 0$. So we don't have uniqueness (no matter what f and g are).