PDE Lecture

Power series solutions

May 19

Power series solutions, Evans 4.6

From last week

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Cauchy problem for kth order quasilinear equation:

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\ldots,u,x)D^\alpha u + a_0(D^{k-1}u,\ldots,u,x) = 0 \quad \text{in } U \qquad (1)$$

$$u = g_0, \quad \frac{\partial u}{\partial \nu} = g_1, \quad \ldots, \quad \frac{\partial^{k-1}u}{\partial \nu^{k-1}} = g_{k-1} \quad \text{on } \Gamma, \qquad (2)$$

$a_\alpha \in C^\infty(U)$.

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$g_0,\ldots,g_{k-1} : \Gamma \to \mathbb{R}$ <u>Cauchy data</u>.

Definition

The surface Γ is <u>noncharacteristic</u> for the PDE (1) provided

$$\sum_{|\alpha|=k} a_\alpha \nu^\alpha \neq 0 \quad \text{on } \Gamma.$$

Theorem (Cauchy-Kovalevskaya Theorem, v. 1)

Let Γ*,* a_α *and* g_k *be analytic near* $x^0 \in \Gamma$ *and assume that* Γ *is noncharacteristic for* (1)*. Then* $\exists$ *unique analytic solution* u *to the Cauchy problem* (1)*,* (2) *near* x^0*.*

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The theorem is proved by transforming (1), (2) to

$$\begin{cases} \mathbf{u}_{x_n} = \sum_{j=1}^{n-1} \mathbf{B}_j(\mathbf{u}, x')\mathbf{u}_{x_j} + \mathbf{c}(\mathbf{u}, x'), & |x| < r \\ \mathbf{u} = 0, & |x'| < r, x_n = 0 \end{cases} \tag{3}$$

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Assume $\{\mathbf{B}_j\}$ *and* $\mathbf{c}$ *are real analytic. There exists* $r > 0$ *and a unique real analytic function* $\mathbf{u} = \sum_\alpha \mathbf{u}_\alpha x^\alpha$ *solving* (3)*.*

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Today: something about the proof.

Real analytic functions

Definition

$f\colon \mathbb{R}^n \to \mathbb{R}$ is (real) analytic near x_0 if there exists $r > 0$ and constants $\{f_\alpha\}$ such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha, \quad |x - x_0| < r.$$

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- Maximal region of convergence is more complicated for $n > 1$, e.g. $\sum_k (x_1 x_2)^k$ converges abs. for $|x_1 x_2| < 1$.
- If the series converges absolutely for some x it will converge absolutely for y on the ray between x_0 and x, but not necessarily for $|y - x_0| < |x - x_0|$[1]

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Example

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&= \sum_{\alpha} \frac{|\alpha|!}{r^{|\alpha|} \alpha!} x^\alpha
\end{aligned}
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Abs. conv. for $|x| < r/\sqrt{n}$ (so that $|x_1 + \cdots + x_n| < r$)

Majorants

Definition

If $f = \sum_\alpha f_\alpha x^\alpha$, $g = \sum_\alpha g_\alpha x^\alpha$ are power series, we say that g <u>majorizes</u> f, written $g \gg f$, if $g_\alpha \geq |f_\alpha|$ $\forall \alpha$.

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Lemma

1. *If $g \gg f$ and g converges for $|x| < r$, then so does f.*
2. *If f converges for $|x| < r$ and $0 < s\sqrt{n} < r$, then f has an 'explicit' majorant for $|x| < s/\sqrt{n}$.*

Proof

1. $\sum_\alpha |f_\alpha x^\alpha| \leq \sum_\alpha g_\alpha |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} < \infty$.

2. Let s be as in the statement of the lemma and set $y := s(1,\dots,1)$. Then $|y| = s\sqrt{n} < r$ and hence $\sum_\alpha f_\alpha y^\alpha$ converges. Therefore $\exists C$ s.t.

$$|f_\alpha y^\alpha| \leq C \quad \forall \alpha.$$

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Thus

$$|f_\alpha| \leq \frac{C}{y_1^{\alpha_1}\cdots y_n^{\alpha_n}} = \frac{C}{s^{|\alpha|}} \leq C\frac{|\alpha|!}{s^{|\alpha|}\alpha!}$$

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But then

$$g(x) := \frac{Cs}{s-(x_1+\cdots+x_n)} = C\sum_\alpha \frac{|\alpha|!}{s^{|\alpha|}\alpha!}x^\alpha$$

majorizes f for $|x| < s/\sqrt{n}$. □

Proof of Cauchy-Kovalevskaya, v. 2

For notational simplicity, consider the case $m = 1$ ($\mathbf{u} = u$ scalar-valued). Then

$$\begin{cases} u_{x_n} = \sum_{j=1}^{n-1} b_j(u,x')u_{x_j} + c(u,x'), & |x| < r \\ u = 0, & |x'| < r, x_n = 0 \end{cases} \tag{4}$$

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Write $u = \sum_\alpha u_\alpha x^\alpha$, $|x| < r$ for some r to be found,

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Differentiate PDE w.r.t. x_i, $1 \leq i \leq n-1$:

$$u_{x_n x_i} = \sum_{j=1}^{n-1} \left(b_j u_{x_i x_j} + b_{j,x_i} u_{x_j} + b_{j,z} u_{x_i} u_{x_j} \right) + c_{x_i} + c_z u_{x_i}.$$

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Gives $u_{x_n x_i}(0) = c_{x_i}(0,0)$

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$$D^{\alpha}u(0) = D^{\alpha'}c(0,0)$$

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In general, by differentiating the equation several times with respect to x_n:

$$u_{\alpha} = \frac{D^{\alpha}u(0)}{\alpha!} = q_{\alpha}(\cdots, b_{j,\gamma,\delta}, \ldots, c_{\gamma,\delta}, \ldots, u_{\beta}, \ldots),$$

q_{α} polynomial with *nonnegative coefficients* and $\beta_n \leq \alpha_n - 1$.

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q_{α} polynomial with *nonnegative coefficients* and $\beta_n \le \alpha_n - 1$.

Suppose $b_j^* \gg b_j$, $c^* \gg c$, with

$$b_j^* := \sum_{\gamma,\delta} b_{j,\gamma,\delta}^* z^{\gamma} x^{\delta} \quad \text{and} \quad c^* := \sum_{\gamma,\delta} c_{\gamma,\delta}^* z^{\gamma} x^{\delta}, \quad |z| + |x'| < s$$

and consider the new BVP

$$\begin{cases} u_{x_n}^* = \sum_{j=1}^{n-1} b_j^*(u^*, x')u_{x_j}^* + c^*(u^*, x'), & |x| < r \\ u^* = 0, & |x'| < r, x_n = 0. \end{cases} \tag{5}$$

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$$\begin{aligned} |u_\alpha| &= |q_\alpha(\ldots, b_{j,\gamma,\delta}, \ldots, c_{\gamma,\delta}, \ldots, u_\beta, \ldots)| \\ &\leq q_\alpha(\ldots, |b_{j,\gamma,\delta}|, \ldots, |c_{\gamma,\delta}|, \ldots, |u_\beta|, \ldots) && \text{nonneg. coeff.} \\ &\leq q_\alpha(\ldots, b^*_{j,\gamma,\delta}, \ldots, c^*_{\gamma,\delta}, \ldots, u^*_\beta, \ldots) && \beta_n \leq \alpha_n - 1 \\ &= u^*_\alpha \end{aligned}$$

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Choose

$$b^*_j = c^* := \frac{Cr}{r - (x_1 + \ldots + x_{n-1}) - z}$$

with C suff. large, $r > 0$ suff. small and $|x'| + |z| < r$.

We then get

$$\begin{cases} u^*_{x_n} = \dfrac{Cr}{r-(x_1+\ldots+x_{n-1})-u^*}\left(\displaystyle\sum_{j=1}^{n-1} u^*_{x_j}+1\right), & |x|<r \\ u^*=0, & |x'|<r, x_n=0. \end{cases} \tag{6}$$

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Look for solution $u^* = v^*(x_1+\cdots+x_n, x_n)$. Gives

$$\begin{cases} v^*_t = \dfrac{Cr}{r-s-v^*}(1+(n-1)v^*_s), & \\ v^* = 0, & t=0. \end{cases}$$

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$$v^*(s,t) = \frac{1}{n}(r-s-[(r-s)^2-2nCrt]^{1/2})$$

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Hence

$$u^* = \frac{1}{n}(r-(x_1+\cdots+x_{n-1})-[(r-(x_1+\cdots+x_{n-1}))^2-2nCrx_n]^{1/2}).$$

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Power series of u_{x_n} and $\sum_{j=1}^{n-1} b_j(u,x)u_{x_j} + c(u,x)$ agree at $x = 0 \Rightarrow$ they agree for $|x| < r$.