# PDE Lecture

Conservation laws, Power series solutions

May 12

# Conservation laws, Evans 3.4

# Scalar conservation law

One-dimensional case, $F \in C^2(\mathbb{R})$

$$\begin{cases} u_t + (F(u))_x = 0, & x \in \mathbb{R}, t > 0 \\ u = g, & x \in \mathbb{R}, t = 0. \end{cases} \tag{1}$$

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Characteristic equations:

$$\begin{cases} \dot{x}(s) = F'(z(s)) \\ \dot{t}(s) = 1 \\ \dot{z}(s) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x(s) = x^0 + sF'(g(x^0)) \\ t = s \ (+0) \\ z(s) = z^0 = g(x^0) \end{cases}$$

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Implicit formula

$$u(x,t) = g(x^0(x,t)) = g(x - tF'(g(x^0))) = g(x - tF'(u(x,t)))$$

Assume $F$ strictly convex (e.g. $F(u) = u^2/2$), so that $F'$ is strictly increasing.

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### Definition

$u \in L^\infty(\mathbb{R} \times (0,\infty))$ is an integral (or weak) solution of (1) if

$$\int_0^\infty \int_{-\infty}^\infty (uw_t + F(u)w_x)\, dx\, dt + \int_{-\infty}^\infty g w\, dx\Big|_{t=0} = 0$$

for all $w \in C_c^\infty(\mathbb{R} \times [0,\infty))$.

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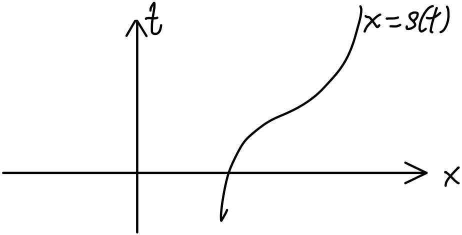



Then $u$ is an integral solution iff it satisfies the problem classically on either side and

$$[[F(u)]] = \sigma[[u]],$$

where

$$[[u]] = u_\ell - u_r, \quad [[F(u)]] = F(u_\ell) - F(u_r), \quad \sigma = \dot{s}.$$

## Example

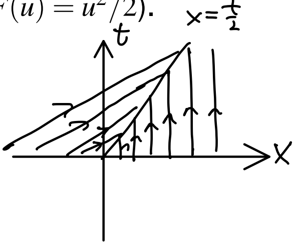
For Burgers' equation $u_t + uu_x = 0$ with

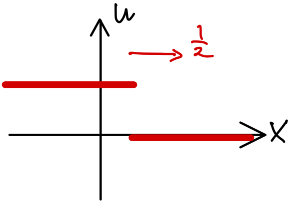
$$g(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

we found the weak solution

$$u(x,t) = \begin{cases} 1, & x < \frac{1}{2}t \\ 0, & x > \frac{1}{2}t \end{cases}.$$

$(F(u) = u^2/2)$.



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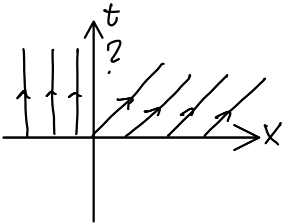


What should the solution look like in the empty wedge?

One possibility:

$$u(x,t) = \begin{cases} 0, & x < \frac{1}{2}t \\ 1, & x > \frac{1}{2}t \end{cases}.$$

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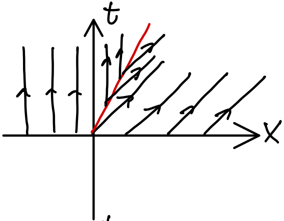


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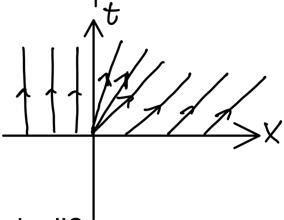
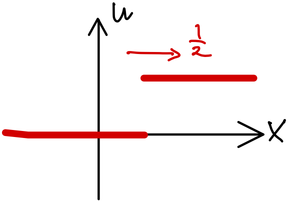
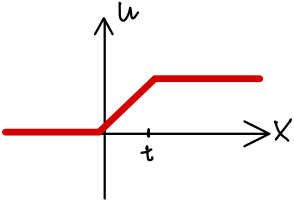



Both are integral solutions! Which one is 'physical'?



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If $F$ is uniformly convex ($F'' \geq \theta > 0$), equivalent to

$$u_\ell > u_r$$

along any shock curve (exercise!).

# Example

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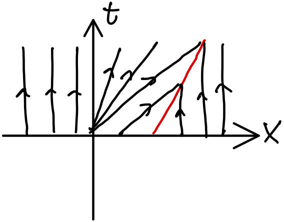


For $0 \le t \le 2$:

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 < x < t \\ 1, & t < x < 1 + \frac{t}{2} \\ 0 & x > 1 + \frac{t}{2} \end{cases}.$$

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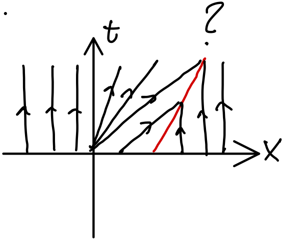


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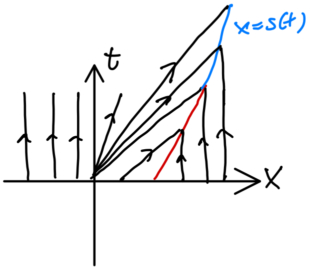
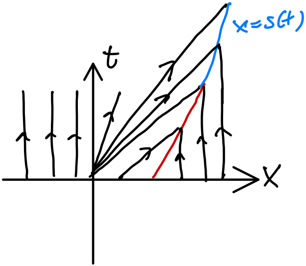
What happens when the rarefaction wave meets the shock wave at $t = 2$?




Expect shock to continue along curve $x = s(t)$, with $u = x/t$ to the left, $u = 0$ to the right.

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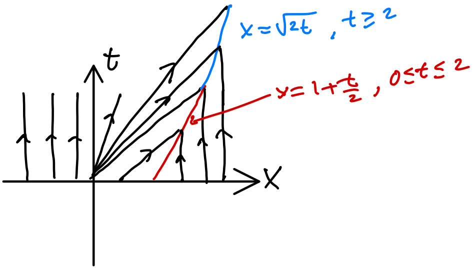
Rankine-Hugoniot:

$$\dot{s}(t)\frac{s(t)}{t} = \sigma[[u]] = [[F(u)]] = \frac{1}{2}\left(\frac{s(t)}{t}\right)^2$$

$$\Rightarrow \dot{s}(t) = \frac{s(t)}{2t}$$

$$\underset{s(2)=2}{\Rightarrow} s(t) = \sqrt{2t}, \quad t \geq 2$$




$$u(x,t) = \begin{cases} 0, & x < 0 \\ \frac{x}{t}, & 0 < x < \sqrt{2t}\,, \quad t \geq 2. \\ 0 & x > \sqrt{2t} \end{cases}$$

More flexible entropy condition ($F$ convex):

$$u(x+z,t) - u(x,t) \leq C(1+\tfrac{1}{t})z,$$

for some $C \geq 0$ and a.e. $x, z \in \mathbb{R}$, $t > 0$ with $z > 0$.

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Under this condition, one can prove the uniqueness (and existence) of solutions.

Evans, 3.4.2–3.4.3

# Power series solutions, Evans 4.6

# Noncharacteristic surfaces

$k$th order quasilinear equation

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\ldots,u,x)D^\alpha u + a_0(D^{k-1}u,\ldots,u,x) = 0 \qquad (2)$$

in $U \subset \mathbb{R}^n$, open. $a_\alpha \in C^\infty(U)$.

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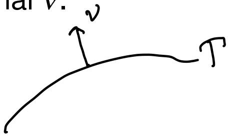

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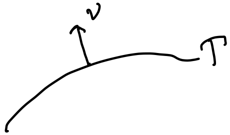
$j$th normal derivative of $u$ at $x^0 \in \Gamma$:

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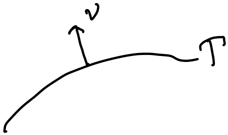


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Cauchy problem: Solve (2) subject to

$$u = g_0, \quad \frac{\partial u}{\partial \nu} = g_1, \quad \ldots, \quad \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = g_{k-1} \quad \text{on } \Gamma, \qquad (3)$$

$g_0,\ldots,g_{k-1} : \Gamma \to \mathbb{R}$ Cauchy data.

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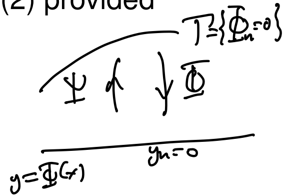


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Idea: Change of variables used to reduce to the case when $\Gamma = \{x_n = 0\}$.
We discuss the proof in this case.

**Proof in the flat case**

Cauchy conditions:

$$u = g_0, \quad \frac{\partial u}{\partial x_n} = g_1, \quad \ldots, \quad \frac{\partial^{k-1} u}{\partial x_n^{k-1}} = g_{k-1} \quad \text{on } \Gamma := \{x_n = 0\}.$$

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Differentiation gives $\frac{\partial u}{\partial x_j} = \frac{\partial g_0}{\partial x_j}$, $1 \leq j \leq n-1$, while $\frac{\partial u}{\partial x_n} = g_1$.
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Similarly, $\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial^2 g_0}{\partial x_j \partial x_k}$, $1 \leq j,k \leq n-1$, while $\frac{\partial^2 u}{\partial x_j \partial x_n} = \frac{\partial g_1}{\partial x_j}$ and $\frac{\partial^2 u}{\partial x_n^2} = g_2$. Hence $D^2 u$ is determined along $\Gamma$.

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Works up until $D^{k-1} u$. For $D^k u$, we can't determine $\frac{\partial^k u}{\partial x_n^k}$ this way.

We use the equation instead:

$$\frac{\partial^k u}{\partial x_n^k} = -\frac{1}{a_{(0,\ldots,0,k)}} \left[ \sum_{\substack{|\alpha|=k \\ \alpha \neq (0,\ldots,0,k)}} a_\alpha D^\alpha u + a_0 \right]$$

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Can be computed by differentiating the PDE (2) w.r.t. $x_n$.
Induction gives all derivatives. □

# Real analytic functions

## Definition

$f : \mathbb{R}^n \to \mathbb{R}$ is (real) analytic near $x_0$ if there exists $r > 0$ and constants $\{f_\alpha\}$ such that

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$f$ analytic $\Rightarrow f \in C^\infty$ near $x_0$ and

$$f_\alpha = \frac{D^\alpha f(x_0)}{\alpha!}.$$

## Theorem (Cauchy-Kovalevskaya Theorem, v. 1)

*Let $\Gamma$, $a_\alpha$ and $g_k$ be analytic near $x^0 \in \Gamma$ and assume that $\Gamma$ is noncharacteristic for* (2)*. Then $\exists$ unique analytic solution $u$ to the Cauchy problem* (2), (3) *near $x^0$.*

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Step 1: Using an analytic change of variables, we can reduce to the following problem

$$\begin{cases} \sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\ldots,u,x)D^\alpha u \\ \qquad +a_0(D^{k-1}u,\ldots,u,x) = 0, \quad |x|<r \\ u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{k-1}u}{\partial x_n^{k-1}} = 0, \quad |x'|<r, x_n = 0, \end{cases} \tag{4}$$

for some $r>0$ to be found.

Step 2: Reduce to a first-order system by introducing

$$\mathbf{u} := \left(u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \cdots, \frac{\partial^{k-1} u}{\partial x_n^{k-1}}\right)$$

Then $\mathbf{u} \colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{u} = (u^1, \ldots, u^m)$.
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The new system is of the form

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Something about the proof next time.