LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

SOLUTIONS, MAY 15

Evans 3.5

4. **a)** Characteristic equations

$$\begin{cases} \dot{x} = b \\ \dot{t} = 1 \\ \dot{z} = f(x,t). \end{cases}$$

b) We get

$$\begin{cases} x(s) = x^0 + sb \\ t(s) = s\ (+0) \\ z(s) = z^0 + \int_0^s f(x(r),t(r))\,dr = z(0) + \int_0^s f(x^0 + rb, r)\,dr \end{cases}$$

where the parametrisation has been chosen so that the projected characteristic passes through $(x^0, 0)$ when $s = 0$. Evaluating at $s = 0$ and using the initial condition, we get $z^0 = u(x^0, 0) = g(x^0)$. Thus

$$u(x,t) = z(t) = g(x^0) + \int_0^t f(x^0 + rb, r)\,dr.$$

Finally, $x^0 = x - bt$ givs

$$u(x,t) = g(x - bt) + \int_0^t f(x + b(r - t), r)\,dr.$$

5. Solve using characteristics.

a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u$, $u(x_1, 1) = g(x_1)$.

Solution: Characteristic equations

$$\begin{cases} \dot{x}^1 = x^1 \\ \dot{x}^2 = x^2 \\ \dot{z} = 2z \end{cases} \Leftrightarrow \begin{cases} x^1(s) = x_1^0 e^s \\ x^2(s) = x_2^0 e^s \\ z(s) = z^0 e^{2s} \end{cases}$$

Choose parametrisation s.t. $x^2(0) = 1$. Then $x_2^0 = 1$ and x_1^0 is the x_1 coordinate at 'initial time' (when $x^2 = 1$). Moreover, $z(0) = z^0 = g(x_1^0)$. We thus have

$$\begin{cases} x^1(s) = x_1^0 e^s \\ x^2(s) = e^s \\ z(s) = g(x_1^0) e^{2s} \end{cases}$$

Please, turn over!

Want to express $u = z$ in terms of (x_1, x_2). We have

$$e^s = x_2 \quad \Leftrightarrow \quad s = \log x_2.$$

Then

$$x_1^0 e^s = x_1 \quad \Leftrightarrow \quad x_1^0 x_2 = x_1 \quad x_1^0 = \frac{x_1}{x_2}.$$

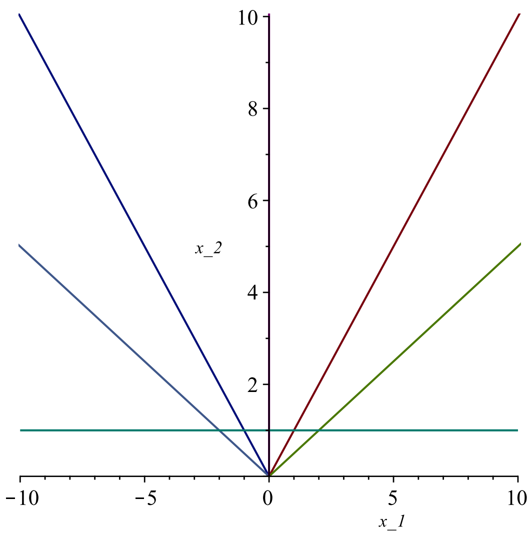
Moreover,

$$u(x_1, x_2) = z(s) = g(x_1^0) e^{2s} = g(\tfrac{x_1}{x_2}) x_2^2, \quad x_2 \neq 0$$

Test:

$$x_1 u_{x_1} + x_2 u_{x_2} = x_1 \left(\tfrac{1}{x_2}\right) g'(\tfrac{x_1}{x_2}) x_2^2 + x_2 \left(-\tfrac{x_1}{x_2^2}\right) g'(\tfrac{x_1}{x_2}) + 2g(\tfrac{x_1}{x_2}) x_2^2 = 2u.$$

Note that the (projected) characteristics are straight rays from the origin. Thus the solution is uniquely determined by the 'initial condition' in the upper half-plane. The solution formula makes sense also in the lower half-plane, but since the domain $\{(x_1, x_2) \colon x_2 \neq 0\}$ is not connected we could equally well set $u = 0$ in the lower half-plane. Therefore, the upper half-plane is the natural maximal domain of the solution.



b) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$, $u(x_1, x_2, 0) = g(x_1, x_2)$.

Solution: Characteristic equations

$$\begin{cases} \dot{x}^1 = x^1 \\ \dot{x}^2 = 2x^2 \\ \dot{x}^3 = 1 \\ \dot{z} = 3z \end{cases} \Leftrightarrow \begin{cases} x^1(s) = x_1^0 e^s \\ x^2(s) = x_2^0 e^{2s} \\ x^3(s) = s\ (+0) \\ z(s) = z^0 e^{3s} \end{cases}$$

where again the parametrisation is chosen such that $x^3 = 0$ when $s = 0$. Then $z(0) = z^0 = g(x_1^0, x_2^0)$ and we have

$$\begin{cases} x^1(s) = x_1^0 e^s \\ x^2(s) = x_2^0 e^{2s} \\ x^3(s) = s \\ z(s) = g(x_1^0, x_2^0) e^{3s} \end{cases}$$

Again, we want to write z in terms of x and thus solve $(x^1, x^2, x^3) = (x_1, x_2, x_3)$ for s, x_1^0 and x_2^0:

$$\begin{cases} x_1^0 = x_1 e^{-x_3} \\ x_2^0 = x_2 e^{-2x_3} \\ s = x_3 \end{cases}$$

Hence

$$u(x_1, x_2, x_3) = z(s) = g(x_1^0, x_2^0) e^{3s} = g(x_1 e^{-x_3}, x_2 e^{-2x_3}) e^{3x_3}.$$

Again, one can directly verify that this is a solution (assuming that $g \in C^1$) and the natural domain is $\mathbb{R}^3$.

c) $u u_{x_1} + u_{x_2} = 1$, $u(x_1, x_1) = \frac{1}{2} x_1$.

Solution: Characteristic equations

$$\begin{cases} \dot{x}^1 = z \\ \dot{x}^2 = 1 \\ \dot{z} = 1 \end{cases} \Leftrightarrow \begin{cases} \dot{x}^1(s) = s + z^0 \\ x^2(s) = s + x_2^0 \\ z(s) = s + z^0 \end{cases} \Leftrightarrow \begin{cases} x^1(s) = \dfrac{s^2}{2} + z^0 s + x_1^0 \\ x^2(s) = s + x_2^0 \\ z(s) = s + x_3^0 \end{cases}$$

Choose parametrisation such that $x^1(0) = x^2(0)$. This means that $x_1^0 = x_2^0$. Then

$$z^0 = z(0) = u(x_1^0, x_1^0) = \tfrac{1}{2} x_1^0$$

We get

$$\begin{cases} x^1(s) = \dfrac{s^2}{2} + \frac{1}{2} x_1^0 s + x_1^0 \\ x^2(s) = s + x_1^0 \\ z(s) = s + \frac{1}{2} x_1^0 \end{cases}$$

$(x^1(s), x^2(s))$ parametrises a parabola passing through the point $(2,2)$ (see figure below).

Please, turn over!

Solving $(x^1, x^2) = (x_1, x_2)$ for s and x_1^0 we get

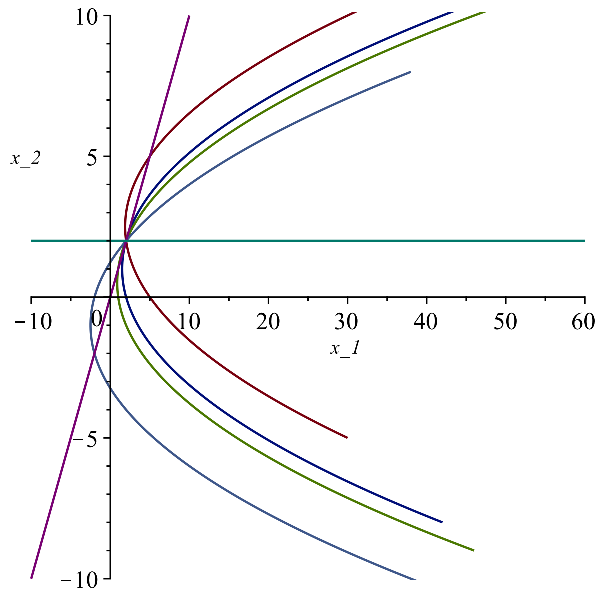
$$\begin{cases} \dfrac{s^2}{2} + \frac{1}{2}x_1^0 s + x_1^0 = x_1 \\ s + x_1^0 = x_2 \end{cases} \Leftrightarrow \begin{cases} \dfrac{s^2}{2} + \frac{1}{2}(x_2 - s)s + x_2 - s = x_1 \\ x_1^0 = x_2 - s \end{cases}$$

$$\Leftrightarrow \begin{cases} (\frac{1}{2}x_2 - 1)s + x_2 = x_1 \\ x_1^0 = x_2 - s \end{cases} \Leftrightarrow \begin{cases} s = 2\dfrac{x_1 - x_2}{x_2 - 2} \\ x_1^0 = x_2 - s \end{cases}$$

Hence

$$u(x_1, x_2) = z(s) = s + \frac{1}{2}x_1^0 = s + \frac{1}{2}(x_2 - s) = \frac{1}{2}s + \frac{1}{2}x_2 = \frac{x_1 - x_2}{x_2 - 2} + \frac{1}{2}x_2, \quad x_2 \neq 2.$$

The point $(2,2)$ is singular and the natural domain is either $\{(x_1, x_2) \colon x_2 > 2\}$ or $\{(x_1, x_2) \colon x_2 < 2\}$. Note that the projected characteristics only cross from one half-plane to the other through the singular point.



8. Show that $u = g(x - tF'(u)) = g(x - tf(u))$ provides an implicit solution for $u_t + (F(u))_x = 0$ $(f(u) = F'(u))$.

Solution: Assume that $1 + tg'(x - tf(u))f'(u) \neq 0$, so that the implicit function theorem applies near (x,t,u). Implicit differentiation of the relation

$$u = g(x - tf(u))$$

gives

$$u_t = -g'(x - tf(u))f(u) - tg'(x - tf(u))f'(u)u_t,$$

so that

$$u_t = -\frac{g'(x - tf(u))f(u)}{1 + tg'(x - tf(u))f'(u)}.$$

Similarly,

$$u_x = g'(x - tf(u)) - tg'(x - tf(u))f'(u)u_x$$

and hence

$$u_x = \frac{g'(x - tf(u))}{1 + tg'(x - tf(u))f'(u)}.$$

It follows that

$$u_t + (F(u))_x = u_t + f(u)u_x = -\frac{g'(x - tf(u))f(u)}{1 + tg'(x - tf(u))f'(u)} + \frac{g'(x - tf(u))f(u)}{1 + tg'(x - tf(u))f'(u)} = 0.$$

Note that the condition $1 + tg'(x - tf(u))f'(u) \neq 0$ is automatically satisfied at initial time $t = 0$. In the Burgers' case, $f(u) = u$, the condition simplifies to $1 + tg'(x - tf(u)) \neq 0$ and this will be satisfied for $0 \leq t < T$, where

$$T = -\frac{1}{\inf g'},$$

assuming $g' < 0$ somewhere. If $g' \geq 0$ the condition is satisfied everywhere. One can check that T is also the first time the characteristics intersect (do this!). Thus until this time, we can use implicit differentiation to verify that u is indeed a solution.

19. First assume that the solution is C^1. Then

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t(x,t)\,dx = -\int_{-\infty}^{\infty} (F(u(x,t))_x\,dx = -[F(u(x,t))]_{x=-\infty}^{\infty} = 0,$$

where the steps are permissible since u has compact support in x. This. proves the claim in that case, since

$$\int_{-\infty}^{\infty} u(x,0)\,dx = \int_{-\infty}^{\infty} g(x)\,dx.$$

In the case of a continuous integral solution, we use the definition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (uv_t + F(u)v_x)\,dx\,dt + \int_{-\infty}^{\infty} gv\,dx|_{t=0} = 0,$$

for each $v \in C^\infty(\mathbb{R} \times [0,\infty))$. Since u is supposed to have compact support in $\mathbb{R} \times [0,T]$ for each $T > 0$, we can take v of the form $\varphi(x)\psi(t)$, where $\varphi = 1$ on the support of $u(\cdot,t)$ for each $t \in [0,T]$ and $\psi(0) = 1$. This gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\psi_t\,dx\,dt + \int_{-\infty}^{\infty} g\,dx = 0.$$

Please, turn over!

Next, we choose $\psi = \psi^\varepsilon(t)$ such that $\psi_t^\varepsilon \to -\delta_T$ (the Dirac delta distribution at $t = T$) as $\varepsilon \to 0$, in the sense of distributions. More specifically, we can pick a smooth function $\eta(t)$ with compact support and integral 1 and set $\psi^\varepsilon(t) = \int_t^\infty \varepsilon^{-1}\eta(\varepsilon^{-1}(s-T))\,ds$. Then ψ^ε is constant when t is outside the support of $\eta(\varepsilon^{-1}(\cdot - T))$. In particular, it is constant on $(-\infty, 0]$ if $\varepsilon > 0$ is sufficiently small and the value is $\int_{-\infty}^\infty \varepsilon^{-1}\eta(\varepsilon^{-1}(s-T))\,ds = \int_{-\infty}^\infty \eta(s)\,ds = 1$. On the other hand, $(\psi^\varepsilon)'(t) = -\varepsilon^{-1}\eta(\varepsilon^{-1}(s-T)) \to -\delta_T$ in the sense of distributions. In fact, the convergence holds even if the test function is only assumed continuous. Using this ψ above and letting $\varepsilon \to 0$ gives

$$-\int_{-\infty}^{\infty} u(x,T)\,dx + \int_{-\infty}^{\infty} g\,dx = 0$$

or, equivalently,

$$\int_{-\infty}^{\infty} u(x,T)\,dx = \int_{-\infty}^{\infty} g\,dx.$$

20. Using the Rankine-Hugoniot and entropy conditions, we find that the solution is given by

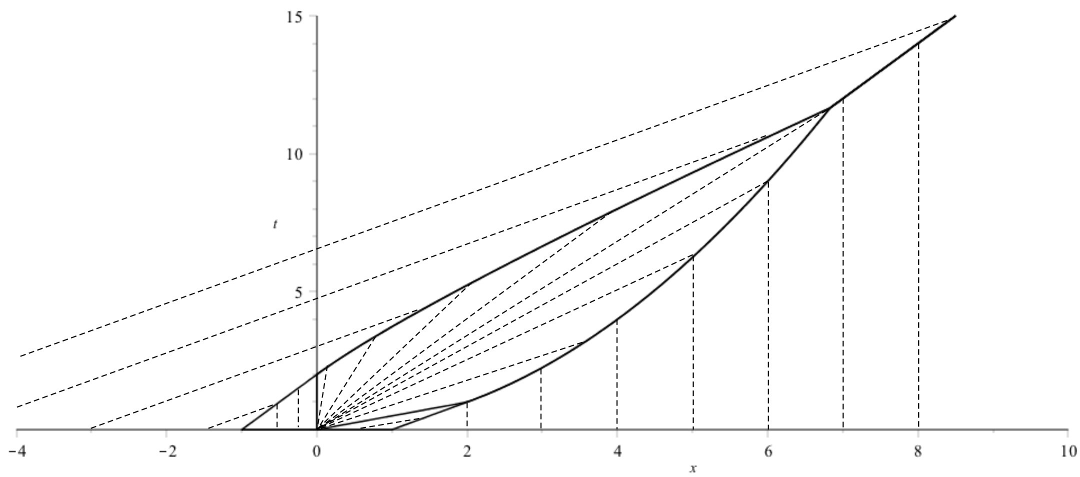
$$u(x,t) = \begin{cases} 1, & x < -1 + \frac{1}{2}t, \\ 0, & -1 + \frac{1}{2}t < x < 0 \\ \frac{x}{t}, & 0 < x < 2t, \\ 2, & 2t < x < 1 + t, \\ 0, & 1 + t < x \end{cases} \qquad 0 < t < 1$$

$$u(x,t) = \begin{cases} 1, & x < -1 + \frac{1}{2}t, \\ 0, & -1 + \frac{1}{2}t < x < 0 \\ \frac{x}{t}, & 0 < x < 2\sqrt{t}, \\ 0, & 2\sqrt{t} < x \end{cases} \qquad 1 < t < 2$$

$$u(x,t) = \begin{cases} 1, & x < t - \sqrt{2t}, \\ \frac{x}{t}, & t - \sqrt{2t} < x < 2\sqrt{t}, \\ 0, & 2\sqrt{t} < x \end{cases} \qquad 2 < t < 6 + 4\sqrt{2}$$

$$u(x,t) = \begin{cases} 1, & x < 1 + \frac{1}{2}t, \\ 0, & 1 + \frac{1}{2}t < x \end{cases} \qquad 6 + 4\sqrt{2} < t.$$

See the characteristics in the figure below.



Evans 4.8

19. We did the first part in the lecture.

To prove the second part, assume that such a solution exists and let

$$u(x,t) = \sum_{m,n} a_{m,n} x^m t^n$$

be the power series expansion near the origin. Plugging this in the equation, and equating coefficients, we get

$$(n+1)a_{m,n+1} = (m+2)(m+1)a_{m+2,n}. \tag{1}$$

Moreover,

$$\sum_m a_{m,0} x^m = \frac{1}{1+x^2} = \sum_k (-1)^k x^{2k}.$$

Thus, $a_{m,0} = 0$ for odd m. Using (1), we obtain, that $a_{m,1}$ also vanishes for odd m (take $n = 0$) and then by induction that the same is true for $a_{m,n}$ for any $n \geq 0$. On the other hand,

$$\begin{aligned}
a_{2k,0} &= (-1)^k \\
a_{2k,1} &= (2k+2)(2k+1)a_{2k+2,0} = (-1)^k(2k+2)(2k+1) \\
a_{2k,2} &= (-1)^k \frac{(2k+4)(2k+3)(2k+2)(2k+1)}{2} \\
&\vdots \\
a_{2k,j} &= (-1)^k \frac{(2k+2j)(2k+2j-1)\cdots(2k+2)(2k+1)}{j!}.
\end{aligned}$$

Thus u is given by

$$u(x,t) = \sum_{k,j} a_{2k,j} x^{2k} t^j.$$

In particular, taking $x = 0$, $t > 0$, we get

$$u(0,t) = \sum_j a_{0,j} t^j = \sum_j \frac{(2j)!}{j!} t^j,$$

which diverges by the ratio test.