PDE Lecture

Nonlinear 1st order PDE and the method of characteristics

May 5

General form

$$F(Du, u, x) = 0$$

$F\colon \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$ given, $U \subset \mathbb{R}^n$ open, $u\colon \overline{U} \to \mathbb{R}$.

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  Quasilinear.

- Conservation law:

$$u_t + \operatorname{div} \mathbf{F}(u) = 0, \quad u = u(x,t), x \in \mathbb{R}^n; \quad \mathbf{F}\colon \mathbb{R} \to \mathbb{R}^n.$$

  Quasilinear.

- Eikonal eq:

$$|Du| = 1 \Leftrightarrow \sum_{i=1}^{n} u_{x_i}^2 = 1, \quad u = u(x), x \in \mathbb{R}^n.$$

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- Hamilton-Jacobi eq:

$$u_t + H(Du) = 0, \quad u = u(x,t), x \in \mathbb{R}^n, \quad H\colon \mathbb{R}^n \to \mathbb{R}.$$

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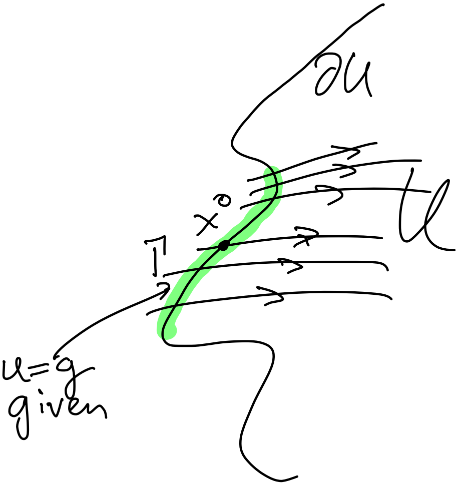
Method of characteristics

<u>Idea:</u> Convert to ODE along characteristics



Linear case

$$\mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0 \tag{1}$$

$\mathbf{b} \in C^1(\overline{U}, \mathbb{R}^n)$, $c \in C(\overline{U}, \mathbb{R})$.

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If u is a solution, then

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Solution

$$z(s) = z(0)e^{-\int_0^s c(\mathbf{x}(r))\,dr}.$$

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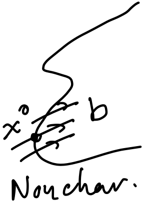

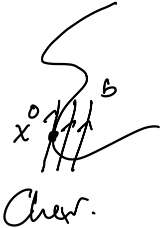

Given a noncharacteristic $x^0 \in \partial U$ and a function g def. in a nbh. Γ of x^0, we can locally solve (2) near x^0, with $\mathbf{x}(0) = \mathbf{x}(0; y)$ parametrizing Γ, $y \in \mathbb{R}^{n-1}$.

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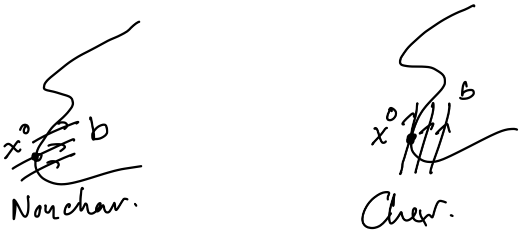

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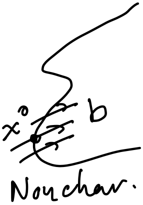

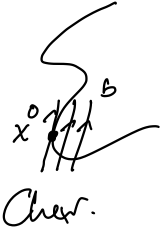

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Then $u(\mathbf{x}(s;y)) = z(s;y) = g(\mathbf{x}(0;y))e^{-\int_0^s c(\mathbf{x}(r))\,dr}$.

$\mathbf{x}(s;y)$ covers a nbh. of x^0 since $\mathbf{b}(x^0) \cdot \nu(x^0) \neq 0$ (inverse function thm.)

Example

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } x_1 > 0, x_2 > 0, \\ u = g & \text{in } x_1 > 0, x_2 = 0. \end{cases}$$

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$$\begin{cases} \dot{x}^1 = -x^2 \\ \dot{x}^2 = x^1 \\ \dot{z} = z \end{cases} \quad \Leftrightarrow \quad \begin{cases} x^1(s) = a\cos s + b\sin s \\ x^2(s) = a\sin s - b\cos s \\ z(s) = z^0 e^s \end{cases}$$

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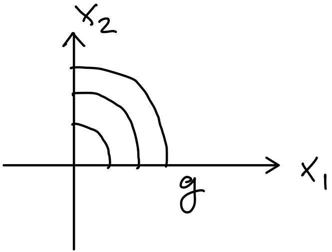
ICs

$$\begin{cases} x^1(0) = a > 0 \\ x^2(0) = -b = 0 \\ z(0) = z^0 = g(a) \end{cases} \Rightarrow \begin{cases} x^1(s) = a\cos s \\ x^2(s) = a\sin s \\ z(s) = g(a)e^s \end{cases}$$



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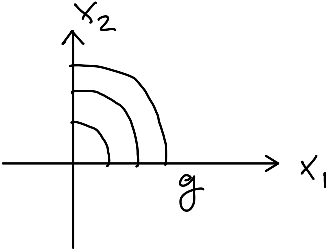

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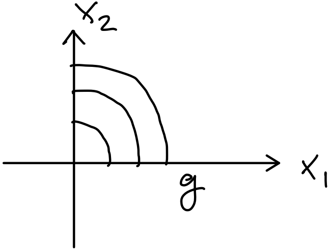
$$\begin{cases} a\cos s = x_1 \\ a\sin s = x_2 \end{cases} \underset{0\le s\le \frac{\pi}{2}}{\Leftrightarrow} \begin{cases} a = \sqrt{x_1^2 + x_2^2} \\ s = \arctan\left(\frac{x_2}{x_1}\right) \end{cases}$$



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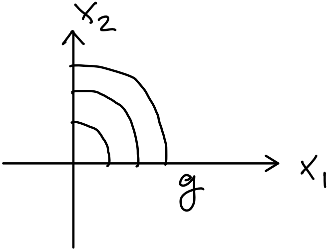

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Can check directly that this is a solution if $g \in C^1((0,\infty))$.

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<u>Remark:</u>
Picard-Lindelöf $\Rightarrow$ different trajectories $s\mapsto(\mathbf{x}(s),z(s))$ can't intersect.
The projected characteristics $s\mapsto\mathbf{x}(s)$ may now intersect, however. (Compare autonomous/nonautonomous system of ODE)

Example

Burgers' equation

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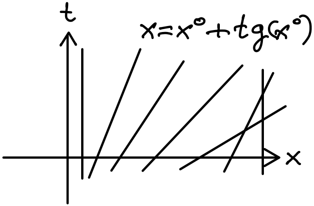
$$\begin{cases} \dot{x}(s) = z(s) \\ \dot{t}(s) = 1 \\ \dot{z}(s) = 0 \end{cases} \Leftrightarrow \begin{cases} x(s) = x^0 + sz^0 = x^0 + sg(x^0) \\ t(s) = s + t_0 = s \\ z(s) = z^0 = g(x^0) \end{cases} \qquad (t_0 = 0)$$



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Expressed in terms of (x,t):

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Crossing (projected) characteristics $\Rightarrow$ solution breaks down!

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Reduce to quasilinear PDE by differentiation:

$$\sum_{j=1}^{n} F_{p_j}(Du, u, x) u_{x_i x_j} + F_z(Du, u, x) u_{x_i} + F_{x_i}(Du, u, x) = 0, \quad i = 1, \dots, n.$$

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Quasilinear system of 1st order PDEs for u, $p=Du$:

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Characteristic equations:

$$\begin{cases} \dot{\mathbf{p}} = -D_z F(\mathbf{p},z,\mathbf{x})\mathbf{p} - D_x F(\mathbf{p},z,\mathbf{x}) \\ \dot{z} = D_p F(\mathbf{p},z,\mathbf{x})\cdot\mathbf{p} \\ \dot{\mathbf{x}} = D_p F(\mathbf{p},z,\mathbf{x}) \end{cases}$$

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Start with (p^0, z^0, x^0) satisfying compatibility condition

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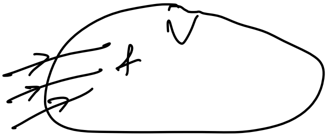
Under these conditions and assuming F, g, and ∂U are smooth, the method of characteristics provides a local solution to the PDE.
See Theorem 2, p. 106.

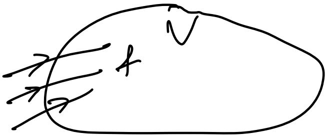
Conservation laws

Recall from first lecture:
$u = u(x,t)$ density of some quantity in $\mathbb{R}^n$ (e.g. chemical concentration)
$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ flux



Recall from first lecture:
$u = u(x,t)$ density of some quantity in $\mathbb{R}^n$ (e.g. chemical concentration)
$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ flux

$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} \mathbf{f}\cdot\nu\,dS = -\int_V \operatorname{div}\mathbf{f}\,dx$$



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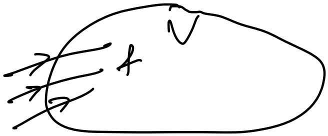
$$u_t + \operatorname{div}\mathbf{f} = 0$$



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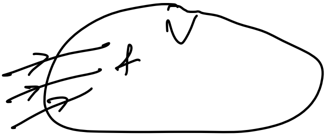
$\mathbf{f} = ?$ Constitutive relation.

Examples: $\mathbf{f} = u\mathbf{b}$ (transport eq.), $\mathbf{f} = -aDu$ heat/diffusion eq.



Scalar conservation law

$\mathbf{f}(x) = \mathbf{F}(u(x))$, $\mathbf{F} \colon \mathbb{R} \to \mathbb{R}^n$

$$\begin{cases} u_t + \mathbf{F}'(u) \cdot Du = 0, & x \in \mathbb{R}, t > 0 \\ u = g, & x \in \mathbb{R}, t = 0. \end{cases} \tag{4}$$

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Characteristic equations:

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{F}'(z(s)) \\ \dot{t}(s) = 1 \\ \dot{z}(s) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \mathbf{x}(s) = x^0 + s\mathbf{F}'(g(x^0)) \\ t = s\ (+0) \\ z(s) = z^0 = g(x^0) \end{cases}$$

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Projected characteristics are straight lines, u is constant along projected characteristics.

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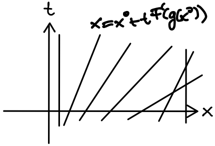

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Implicit formula

$$u(x,t) = g(x^0(x,t)) = g(x - t\mathbf{F}'(g(x^0))) = g(x - t\mathbf{F}'(u(x,t)))$$

Example

$n = 1$, $F(u) = \frac{u^2}{2}$ gives Burgers' equation $u_t + uu_x = 0$

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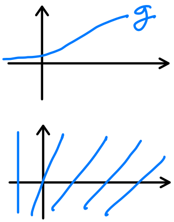
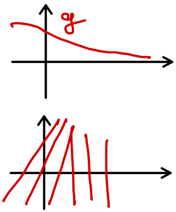
$$1 + tg'(x - tu) \neq 0$$

OK for $t > 0$ if $g'(x) > 0$, $\forall x$. Not OK if $g' < 0$ somewhere.

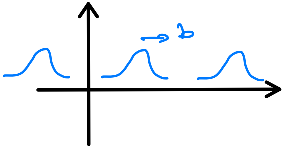




Interpretation:

$b > 0$, $u_t + bu_x = 0 \Rightarrow u = g(x - bt)$.

Wave travelling to the right with speed b.



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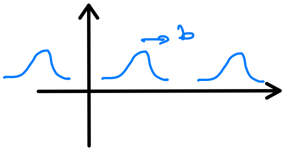

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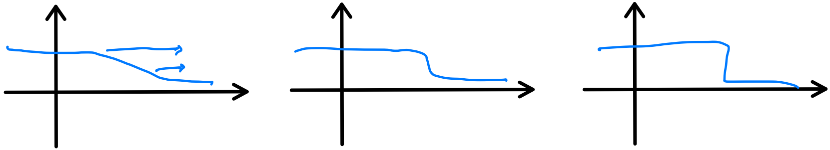
Speed depends on u.



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- Weak solutions

Definition

$u \in L^\infty(\mathbb{R}\times(0,\infty))$ is an integral (or weak) solution of (4) if

$$\int_0^\infty \int_{-\infty}^\infty (uw_t + F(u)w_x)\,dx\,dt + \int_{-\infty}^\infty gw\,dx\Big|_{t=0} = 0$$

for all $w \in C_c^\infty(\mathbb{R}\times[0,\infty))$.

Example (Shock wave for Burgers' eq.)

$u_t + uu_x = 0$

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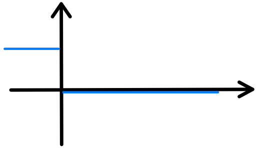

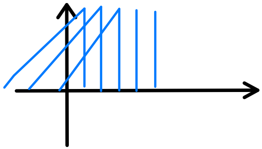

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Ansatz:

$$u(x,t) = \begin{cases} 1, & x < s(t) \\ 0, & x > s(t) \end{cases}$$

$s \in C^1$.

Let $V \subset \mathbb{R} \times (0, \infty)$

$V_\ell = \{(x,t) \in V \colon x < s(t)\}$, $V_r = \{(x,t) \in V \colon x > s(t)\}$

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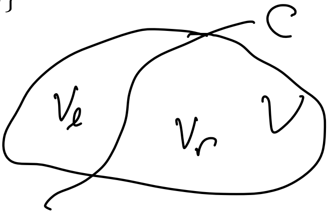

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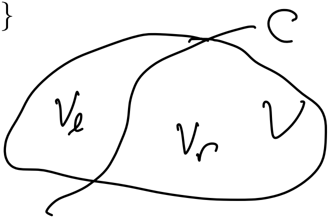

Take $w \in C^\infty$ with compact support in V:

$$0 = \iint_V \left(u w_t + \frac{u^2}{2} w_x \right) dx\, dt$$

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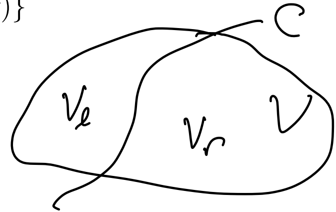

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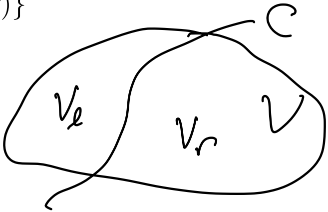

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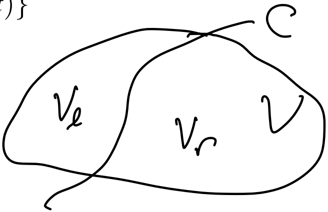

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w arbitrary $\Rightarrow$

$$\nu^2 + \frac{1}{2}\nu^1 = 0 \Leftrightarrow -s'(t) + \frac{1}{2} = 0 \Leftrightarrow s'(t) = \frac{1}{2}.$$

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$$u(x,t) = \begin{cases} 1, & x < \frac{1}{2}t \\ 0, & x > \frac{1}{2}t \end{cases}$$

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For a general F and a general u which is C^1 on either side of C, we get that u is a classical solution on either side and

$$F(u_\ell) - F(u_r) = \dot{s}(u_\ell - u_r)$$

This is called the Rankine-Hugoniot condition.

See Evans pp. 137–139.