LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Written Examination
Partial Differential Equations
Monday, August 22, 2016
08.00-13.00

All functions are assumed to be real-valued throughout the exam.

Note: Only students who are registered or re-registered on the course are allowed to take the exam.

Test results: *Posted Tuesday, August 23, before 17.00. Official viewing of the marked scripts: Wednesday, August 24, 11.30-12.00, in room 508.*

Oral exams: *Thursday, August 25 – Tuesday, August 30. State your preference (day and AM/PM) on the cover sheet of your test – at least two options.*

1. Find a C^2 solution of the wave equation

$$u_{tt} = u_{xx}, \quad x \in (0,\pi),\, t > 0,$$

with boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

and initial conditions

$$u(x,0) = \sin(2x), \quad u_t(x,0) = \sin^3 x.$$

2. Assume that $U \subset \mathbb{R}^n$ is open and bounded and that ∂U is C^1. Let $u \in C^2(\overline{U} \times [0,T])$, $T > 0$, be a solution of the equation

$$u_t - \Delta u + \sum_{i=1}^{n} b_i u_{x_i} + c(x,t)u = 0, \quad x \in U,\, t \in (0,T),$$

with

$$u(x,t) = 0, \quad x \in \partial U,\, t \in [0,T],$$

where b_i, $i = 1,\ldots,n$, are real numbers and $c \in C(\overline{U} \times [0,T])$ with $c(x,t) \geq 0$ for all $(x,t) \in \overline{U} \times [0,T]$. Show that

$$\int_U u^2(x,t)\,dx$$

is a decreasing function of t.

Hint: The divergence theorem might be useful.

3. **a)** Solve the problem

$$xu_x - yu_y = -u, \quad y > 0,$$

with $u(x,1) = \sin x$.

b) Can the solution be extended to $\mathbb{R}^2$ and, if so, is the extension unique (in the class of $C^1(\mathbb{R}^2)$ solutions)?

Please, turn over!

4. Let U be a bounded, open subset of $\mathbb{R}^2$ and let L be the second-order linear partial differential operator defined by $Lu = -u_{x_1x_1} + 2u_{x_1x_2} - 2u_{x_2x_2} + x_1u_{x_1}$.

a) Show that L is uniformly elliptic.

b) Let $u \in C^2(U) \cap C(\overline{U})$ be a solution of the boundary-value problem

$$\begin{aligned} Lu &= u - u^3 \quad \text{in } U, \\ u &= 0 \qquad\quad\; \text{on } \partial U. \end{aligned}$$

Show that $|u| \leq 1$.

5. Let $H^1_{\text{per}}(\mathbb{R}) = \{u \in H^1_{\text{loc}}(\mathbb{R})\colon\, u(x+2\pi) = u(x)\}$ be the Hilbert space of real-valued 2π-periodic functions which are locally in the Sobolev space H^1, equipped with the inner product

$$(u,v)_{H^1_{\text{per}}(\mathbb{R})} = \int_{-\pi}^{\pi} (u(x)v(x) + u'(x)v'(x))\,dx.$$

Define $C^k_{\text{per}}(\mathbb{R})$ and $L^2_{\text{per}}(\mathbb{R})$ similarly.

a) Show that

$$\int_{-\pi}^{\pi} (u(x))^2\,dx \leq \int_{-\pi}^{\pi} (u'(x))^2\,dx$$

for all $u \in H^1_{\text{per}}(\mathbb{R})$ with $\int_{-\pi}^{\pi} u(x)\,dx = 0$.

b) Let $a \in C^1_{\text{per}}(\mathbb{R})$ and $f \in C_{\text{per}}(\mathbb{R})$ and assume that $u \in C^2_{\text{per}}(\mathbb{R})$ is a solution of the equation

$$-(a(x)u')' = f. \tag{1}$$

Show that

$$\int_{-\pi}^{\pi} a(x)u'(x)v'(x)\,dx = \int_{-\pi}^{\pi} f(x)v(x)\,dx \tag{2}$$

for all $v \in C^1_{\text{per}}(\mathbb{R})$.

c) Assume now that $a \in C_{\text{per}}(\mathbb{R})$, $f \in L^2_{\text{per}}(\mathbb{R})$. We say that $u \in H^1_{\text{per}}(\mathbb{R})$ is a weak solution of (1) if (2) holds for all $v \in H^1_{\text{per}}(\mathbb{R})$. Assume that $\min_{x\in\mathbb{R}} a(x) > 0$. Show that the equation

$$-(a(x)u')' = f$$

has a unique weak solution $u \in H^1_{\text{per}}(\mathbb{R})$ with $\int_{-\pi}^{\pi} u(x)\,dx = 0$ for each $f \in L^2_{\text{per}}(\mathbb{R})$ with $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Show also that the condition $\int_{-\pi}^{\pi} f(x)\,dx = 0$ is necessary for the equation to have a weak solution.