Written Examination
Partial Differential Equations
Monday, May 28, 2018
08.00-13.00

LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Throughout the exam, all functions are assumed to be real-valued.

Note: Only students who are registered or re-registered on the course are allowed to take the exam.

Test results: *Posted Tuesday, May 29, before 17.00. Viewing of marked exam scripts: Wednesday, May 30, 11.30-12.00, in room 508.*

Oral exams: *Monday, June 4 – Tuesday, June 5. State your preference (day and AM/PM) on the cover sheet of your test – at least two options.*

1. Find a C^2 solution of the wave equation

$$u_{tt} = u_{xx}$$

for $t > 0$ and $x \in (0, \pi)$, with homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0$$

and initial conditions

$$u(x,0) = 0,$$
$$u_t(x,0) = \cos^2 x.$$

2. Let L be the partial differential operator $L = \partial_x \partial_t - \partial_t$. Show that the function $u\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u(x,t) = \begin{cases} e^x, & x,t > 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a fundamental solution of L, that is, show that $Lu = \delta_0$ in the sense of distributions.

3. Let $g \in C(\mathbb{R}^n)$ be bounded and assume that $u \in C_1^2(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ is a bounded solution of the heat equation $u_t = \Delta u$, for $x \in \mathbb{R}^n$ and $t > 0$, with $u(x,0) = g(x)$. Show that there exists a constant C, independent of g, such that

$$|\partial_{x_j} u(x,t)| \leq \frac{C\|g\|_\infty}{\sqrt{t}},$$

for $x \in \mathbb{R}^n$, $t > 0$ and $j = 1, \ldots, n$, where $\|g\|_\infty = \sup_{x \in \mathbb{R}^n} |g(x)|$.

Hint: Recall that such a solution is unique and can be expressed using the fundamental solution.

Please, turn over!

4. **a)** Solve the nonlinear first-order PDE

$$xu_x - u_y = u^2, \quad y > 0,$$

with $u(x,0) = g(x)$, where $g \in C^1(\mathbb{R})$ is a given function with $g(x) \geq 0$ for all x. Verify that your answer is well-defined on $\mathbb{R} \times [0,\infty)$ and solves the PDE and initial condition.

b) Assume now that g takes negative values (i.e. there is some $x \in \mathbb{R}$ for which $g(x) < 0$). Show then that there is no C^1 solution defined for all $y \geq 0$. Find a maximal set of the form $\mathbb{R} \times [0, y_{\max})$ on which the solution exists, assuming in addition that g is bounded from below.

5. Let L be a second-order uniformly elliptic operator in nondivergence form with continuous coefficients and vanishing zeroth order term, given by $Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i}$. Assume that U is a nonempty, open and bounded domain with C^1 boundary and let ν denote the outward unit normal on ∂U.

a) Show that the boundary value problem

$$\begin{aligned} Lu &= f \quad \text{in } U, \\ \frac{\partial u}{\partial \nu} + u &= g \quad \text{on } \partial U, \end{aligned}$$

has at most one solution of class $C^2(U) \cap C^1(\overline{U})$ for each $f \in C(\overline{U})$ and $g \in C(\partial U)$.

b) Show by means of a counterexample that uniqueness is lost if the boundary condition is changed to

$$\frac{\partial u}{\partial \nu} - u = g \quad \text{on } \partial U$$

(you are free to choose the dimension $n \geq 1$, the domain U and the coefficients a^{ij} and b^i as you like, as long as L is uniformly elliptic).