LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

SOLUTIONS, APRIL 30

Evans 6.6

8. (Solution by Marko Sobak)
Let
$$L = -\sum_{ij} a^{ij}\,\partial^2_{ij}$$
be uniformly elliptic. Suppose that u is smooth and such that $Lu = 0$. We aim to show that
$$\|Du\|_{L^\infty(U)} \le C(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}).$$
Following the hint, put
$$v_\lambda = |Du|^2 + \lambda u^2.$$
We aim to show that $Lv_\lambda \le 0$ in U for large enough λ. We calculate
$$\partial_i v_\lambda = 2\left(\sum_k \partial_k u\,\partial^2_{ik}u + \lambda u\,\partial_i u\right)$$
$$\partial^2_{ij} v_\lambda = 2\left(\sum_k \partial^2_{jk}u\,\partial^2_{ik}u + \sum_k \partial_k u\,\partial^3_{ijk}u + \lambda\,\partial_j u\,\partial_i u + \lambda\,u\,\partial^2_{ij}u\right).$$
Hence
$$Lv_\lambda = -2\left(\sum_{ijk} a^{ij}\,\partial^2_{jk}u\,\partial^2_{ik}u + \sum_{ijk} a^{ij}\,\partial_k u\,\partial^3_{ijk}u + \lambda\sum_{ij} a^{ij}\,\partial_j u\,\partial_i u\right),$$
where the last term is not present since $Lu = 0$. Note that if we define the vectors $w^k = \partial_k Du$, then we see by uniform ellipticity that the first term satisfies
$$\sum_{ijk} a^{ij} w^k_i w^k_j \ge C\sum_k |w^k|^2 = C|D^2u|^2.$$
For the second term, we observe that
$$0 = \partial_k Lu = -\sum_{ij} \partial_k a^{ij}\,\partial^2_{ij}u - \sum_{ij} a^{ij}\,\partial^3_{ijk}u.$$
Thus
$$\sum_{ijk} a^{ij}\,\partial_k u\,\partial^3_{ijk}u = -\sum_{ijk} \partial_k a^{ij}\,\partial_k u\,\partial^2_{ij}u.$$

Please, turn over!

Since the derivatives $\partial_k a^{ij}$ are bounded by assumption, we get

$$\sum_{ijk} \partial_k a^{ij}\, \partial_k u\, \partial_{ij}^2 u \leq A \sum_k |\partial_k u| \sum_{ij} |\partial_{ij}^2 u| \leq A' |Du||D^2 u|.$$

For the final term we note that uniform ellipticity again gives

$$\sum_{ij} a^{ij}\, \partial_j u\, \partial_i u \geq C|Du|^2$$

for non-constant u. Combining these findings we see that

$$\begin{aligned} -Lv_\lambda &\geq C(|D^2u|^2 + \lambda |Du|^2) - A'|Du||D^2u| \\ &= C(|D^2u|^2 - 2A''|Du||D^2u| + \lambda |Du|^2) \\ &= C(|D^2u| - A''|Du|)^2 + (C\lambda - A''^2)|Du|^2, \end{aligned}$$

which is clearly non-negative if $\lambda \geq A''^2/C$. Hence $Lv_\lambda \leq 0$, so that Theorem 1 in 6.4.1 in Evans implies that

$$\|v_\lambda\|_{L^\infty(U)} = \|v_\lambda\|_{L^\infty(\partial U)}.$$

Hence

$$\begin{aligned} \|Du\|_{L^\infty(U)} &\leq \|\sqrt{v_\lambda}\|_{L^\infty(U)} = \|\sqrt{v_\lambda}\|_{L^\infty(\partial U)} \\ &\leq \sqrt{\||Du|^2\|_{L^\infty(\partial U)} + \lambda \||u|^2\|_{L^\infty(\partial U)}} \\ &\leq \lambda'(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}). \end{aligned}$$

10. (Solution by Alex Bergman)

Let $U \subset \mathbb{R}^n$ be open, bounded with smooth boundary ∂U. Suppose also U is connected. Suppose u is a smooth solution of

$$\begin{cases} -\Delta u = 0, \text{ in } U \\ \frac{\partial u}{\partial \nu} = 0, \text{ in } \partial U \end{cases}.$$

Show $u \equiv C$ for some constant C.

1. Using an energy method.

   Proof. By Green's formula

   $$\int_U |Du|^2 dx = -\int_U u\Delta u dx + \int_{\partial U} \frac{\partial u}{\partial \nu} u dS = 0.$$

   So $Du \equiv 0$. Since U is connected we have $u \equiv C$ for some constant C. □

2. Using a maximum principle argument.

   Proof. Suppose the maximum is not achieved in the interior, i.e. there exists some $x_0 \in \partial U$, such that $u(x_0) > u(x)$ for all $x \in U$. Then the Hopf lemma (∂U is smooth so in particular C^2 and thus has the interior ball condition) implies $\frac{\partial u}{\partial \nu}(x_0) > 0$ which is a contradiction. Thus the maximum is achieved in U, since U is connected we have $u \equiv C$ by the strong maximum principle. □

12. (Solution by Filip Jonsson Kling)

Problem:

We say that the uniformly elliptic operator

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

satisfies the weak maximum principle if for all $u \in C^2(U) \cap C(\bar{U})$

$$\begin{cases} Lu \leq 0 & \text{in } U \\ u \leq 0 & \text{in } \partial U \end{cases}$$

implies that $u \leq 0$ in U.

Suppose that there exists a function $v \in C^2(U) \cap C(\bar{U})$ such that $Lv \geq 0$ in U and $v > 0$ on $\bar{U}$. Show that L satisfies the weak maximum principle.

Solution:

Pick u as above and define $w = u/v$. This is well defined on U since $v > 0$ on $\bar{U}$. Then

$$(v^2 w_{x_i})_{x_j} = (u_{x_i} v - v_{x_i} u)_{x_j} = u_{x_i x_j} v + u_{x_i} v_{x_j} - v_{x_i x_j} u - v_{x_i} u_{x_j}.$$

We want to find an elliptic operator M such that $Mw \leq 0$ in $\{u > 0\}$ and M has no zeroth-order term. In the light of the above calculation we try

$$\begin{aligned}
Mw &= -\sum_{i,j=1}^{n} a^{ij} (v^2 w_{x_i})_{x_j} \\
&= -\sum_{i,j=1}^{n} a^{ij} \left(u_{x_i x_j} v + u_{x_i} v_{x_j} - v_{x_i x_j} u - v_{x_i} u_{x_j} \right) \\
&= -\sum_{i,j=1}^{n} a^{ij} (u_{x_i} v_{x_j} - v_{x_i} u_{x_j}) + v \left(-\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} \right) - u \left(-\sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} \right) \\
&= 0 + v \left(Lu - \sum_{i=1}^{n} b^i u_{x_i} - cu \right) - u \left(Lv - \sum_{i=1}^{n} b^i v_{x_i} - cv \right) \\
&= vLu - uLv + u \sum_{i=1}^{n} b^i v_{x_i} - v \sum_{i=1}^{n} b^i u_{x_i} \\
&= vLu - uLv - \sum_{i=1}^{n} b^i v^2 w_{x_i}
\end{aligned}$$

where we used that $a^{ij} = a^{ji}$. Now adding that last term as a first order term to our definition of M, we get that

$$Mw = vLu - uLv \leq 0$$

on the set $\{u > 0\}$ since there $uLv \geq 0$ and $vLu \leq 0$. Since M is uniformly elliptic (note that v^2 is bounded in $\bar{U}$ strictly above zero), we may use the weak maximum principle for $c = 0$ to say that

$$\max_{\{u \geq 0\}} w = \max_{\partial \{u \geq 0\}} w = 0$$

since $u \leq 0$ on ∂U and $v > 0$ in $\bar{U}$. Hence $\{u > 0\} = \emptyset$ so $u \leq 0$ in U and thus L satisfies the weak maximum principle.

Please, turn over!

Evans 7.5

7. (Solution by Simon Halvdansson)

   Problem:

   Suppose u is a smooth solution of

$$\begin{cases} u_t - \Delta u + cu = 0 \text{ in } U \times (0,\infty) \\ u = 0 \text{ on } \partial U \times [0,\infty) \\ u = g \text{ on } U \times \{t = 0\} \end{cases}$$

   and $c \geq \gamma > 0$. Prove the pointwise exponential decay estimate

$$|u(x,t)| \leq Ce^{-\gamma t}$$

   Solution:

   Consider the auxiliary function $v(x,t) = e^{\gamma t}u(x,t)$. A simple calculation shows that

$$\begin{cases} u_t = e^{-\gamma t}(-\gamma v + v_t) \\ \Delta u = e^{-\gamma t}\Delta v \\ cu = e^{-\gamma t}cv \end{cases} \implies 0 = u_t - \Delta u + cu = v_t - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0$$

   and so the weak maximum principle yields (see the remark on p. 391)

$$\max_{\bar{U}_T} |v| = \max_{\Gamma_T} |v|$$

   but since $u = v = 0$ on $\partial U \times [0,\infty)$, the above maximum is bounded by $\max_{\overline{U}} |g| =: C$. Now

$$|u(x,t)| = e^{-\gamma t}|v(x,t)| \leq e^{-\gamma t} \max_{\bar{U}_T} |v| \leq Ce^{-\gamma t}$$

   as desired.

8. (Solution by Simon Halvdansson)

   Problem:

   Suppose u is a smooth solution of the PDE from Problem 7, that $g \geq 0$, and that c is bounded. Show $u \geq 0$.

   Solution:

   Using the same auxiliary function $v(x,t) = e^{\lambda t}u(x,t)$ we again have

$$v_t - \Delta v + \underbrace{(c-\lambda)}_{\geq 0} v = 0.$$

   Since $c(x,t)$ is bounded we can choose λ so that the last term is positive by growing or shrinking λ. The maximum principle with $v_t + Lv \geq 0, c \geq 0$ then yields that

$$\min_{\bar{U}_T} v \geq -\max_{\Gamma_T} v^- = -\max_{\overline{U}} g^- = 0$$

   since $v = 0$ on ∂U for all t so $v \geq 0$ which yields $u \geq 0$ since u, v have the same sign.