PDE Lecture

Evolution equations with variable coefficients

April 28

2nd order parabolic equations

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$f\colon U_T \to \mathbb{R}$, $g\colon U \to \mathbb{R}$ given.

$$\begin{cases} u_t + Lu = f \text{ in } U_T, \\ \qquad\quad u = 0 \text{ on } \partial U \times [0, T] \\ \qquad\quad u = g \text{ on } U \times \{t = 0\}. \end{cases} \tag{1}$$

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$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u \tag{2}$$

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Uniformly parabolic if:

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall x \in U \text{ and } \xi \in \mathbb{R}^n$$

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- Zeroth-order term $\rightarrow$ creation/depletion.

Weak solutions and regularity (Evans 7.1.1 & 7.1.2)

Weak solutions

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Also makes sense if $\mathbf{u}'(t) \in H^{-1}(U) = (H_0^1(U))'$ and $(\cdot,\cdot)_{L^2(U)}$ replaced by pairing $\langle \cdot, \cdot \rangle$ between H^{-1} and H_0^1.

Definition

$\mathbf{u} \in L^2(0,T;H_0^1(U))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(U))$ is a *weak solution* of the IBVP (1) if

1. $\langle \mathbf{u}', v \rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v)$ for each $v \in H_0^1(U)$ and a.e. $t \in [0,T]$,
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Theorem

$\exists!$ *weak sol. of* (1)*.*

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$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k$$

so that

$$d_m^k(0) = (g, w_k), \qquad 1 \leq k \leq m$$

and

$$(\mathbf{u}_m', w_k) + B[\mathbf{u}_m, w_k; t]) = (\mathbf{f}, w_k).$$

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Uniqueness by energy estimates and Grönwall.

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Theorem

Assume $g \in C^\infty(\overline{U}), f \in C^\infty(\overline{U}_T)$ and mth order compatibility conditions hold for $m = 0, 1, \ldots$. Then the unique weak solution $u \in C^\infty(\overline{U}_T)$.

Maximum principles (Evans 7.1.4)

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uniformly elliptic, $a^{ij}, b^i, c \in C(\overline{U}_T)$, U open & bdd. $a^{ij} = a^{ji}$ w.l.o.g.
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Theorem (Weak maximum principle with $c \equiv 0$)

Let $U \subset \mathbb{R}^n$ open, bounded. Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ and $c \equiv 0$ in U. If $u_t + Lu \leq 0$ in U_T, then

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Remark: subsolutions $\leftrightarrow$ supersolutions, max $\leftrightarrow$ min.

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where we used that $c \equiv 0$.

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Let $\varepsilon \downarrow 0$. $\square$

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$$u_t + Lu \geq 0 \quad \text{at } (x_0, t_0),$$

as $c(x_0, t_0)u(x_0, t_0) \geq 0$.

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2. If $u_t + Lu \leq 0$, introduce $u^\varepsilon = u - \varepsilon t$ as before and let $\varepsilon \downarrow 0$. We still get $u_t^\varepsilon + Lu^\varepsilon < 0$ since $-\varepsilon c(x,t)t \leq 0$. □

Application: Uniqueness

Theorem

Under the same assumptions on L, $\exists$ at most one solution $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ to the BVP

$$\begin{cases} u_t + Lu = f \text{ in } U_T, \\ \qquad u = 0 \text{ on } \partial U \times [0,T] \\ \qquad u = g \text{ on } U \times \{t = 0\}. \end{cases}$$

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The condition $c \geq 0$ is not needed! See the trick in problem 8.

Harnack's inequality

Elliptic version:

Theorem

Assume $u \geq 0$ *is a* C^2 *sol. of*

$$-\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x) u = 0$$

in U *and suppose* $V \subset\subset U$ *is connected. Then* $\exists$ *constant* $C > 0$ *(indep. of* u*) s.t.*

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in U and suppose $V \subset\subset U$ is connected. Then $\exists$ constant $C > 0$ (indep. of u) s.t.

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The proof is technical, but see Evans 2.2.3f for a non-technical proof for Laplace's equation using the mean-value property.

Parabolic version:

Theorem

Assume $u \geq 0$ is a C_1^2 solution of

$$u_t + Lu = 0$$

in U_T and suppose $V \subset\subset U$ is connected. Then for all $0 < t_1 < t_2 \leq T$, $\exists$ constant $C > 0$ (indep. of u) s.t.

$$\sup_V u(\cdot, t_1) \leq C \inf_V u(\cdot, t_2).$$

The proof is even more technical.

Theorem (Strong max. principle with $c \equiv 0$)

Assume $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ satisfies $u_t + Lu \leq 0$ in U_T, where the equation is uniformly parabolic and $c \equiv 0$. Assume also that U is connected. If $\max_{\overline{U}_T} u = u(x_0, t_0)$, $(x_0, t_0) \in U_T$, then u is constant on U_{t_0}.

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Proof: see Evans.