PDE Lecture

Elliptic equations and maximum principles

April 21

Weak solutions of 2nd order elliptic equations with variable coefficients

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Uniformly elliptic:

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall x \in U \text{ and } \xi \in \mathbb{R}^n$$

$a^{ij} = a^{ji}$

Weak solutions

L of divergence form with a^{ij}, b^i, $c \in L^\infty(U)$
$f \in L^2(U)$

Definition

$u \in H_0^1(U)$ is a weak solution of (1) if

$$B[u,v] = (f,v)_{L^2(U)} \quad \forall v \in H_0^1(U),$$

where

$$B[u,v] := \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx$$

Existence

Theorem (First existence theorem)

Let $a^{ij}, b^i, c \in L^\infty(U)$, with $a^{ij} = a^{ji}$, and assume that L is of divergence form and uniformly elliptic. $\exists \gamma \geq 0$ s.t. $\forall \mu \geq \gamma$ and $f \in L^2(U)$ $\exists$ unique weak sol. $u \in H_0^1(U)$ of

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = 0 \text{ on } \partial U. \end{cases}$$

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Theorem

If in addition $b^i = 0$ for all i, $\exists \mu > 0$ s.t. if $c(x) \geq -\mu$ for a.e. $x \in U$, then $\exists$ unique weak sol. $u \in H_0^1(U)$ of (1) *$\forall f \in L^2(U)$.*

(Problem 6.6.2)

Fredholm alternative

Theorem

Under the same assumptions as in the first existence theorem, precisely one of the following alternatives holds:

1. (1) *has a unique weak solution for each* $f \in L^2(U)$*; or*
2. *there exists a weak solution* $u \not\equiv 0$ *of the homogeneous problem* (1) *with* $f = 0$*.*

For a more precise version, see Evans 6.2.3, Thm 4, p. 321.

Regularity

Evans, 6.3

Theorem

1. *If* $a^{ij}, b^i, c \in C^{m+1}(\overline{U})$, $f \in H^m(U)$ *and* $u \in H_0^1(U)$ *is a weak solution of* (1), *then* $u \in H_{loc}^{m+2}(U)$ *(interior regularity).*
2. *If also* ∂U *is* C^{m+2}, *then* $u \in H^{m+2}(U)$ *(boundary regularity).*

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In particular, $a^{ij}, b^i, c \in C^\infty(\overline{U}), f \in C^\infty(\overline{U})$ and ∂U $C^\infty \Rightarrow u \in C^\infty(\overline{U})$.

Maximum principles (Evans 6.4)

We consider

$$Lu = -\sum_{ij} a^{ij}(x)u_{x_i x_j} + \sum_i b^i(x)u_{x_i} + c(x)u$$

uniformly elliptic, $a^{ij}, b^i, c \in C(\overline{U})$, U open & bdd. $a^{ij} = a^{ji}$ w.l.o.g.

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Let $u \in C^2(U)$.

- u is called a <u>subsolution</u> to L if $Lu \leq 0$ in U.
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Theorem (Weak maximum principle)

Let $U \subset \mathbb{R}^n$ open, bounded. Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \equiv 0$ in U. If u is a subsolution to L, then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

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$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Remark: subsolutions $\leftrightarrow$ supersolutions, max $\leftrightarrow$ min.

Proof

1. Assume $Lu < 0$ & $\exists x_0 \in U$ s.t. $u(x_0) = \max_{\overline{U}} u$. Then
$$Du(x_0) = 0 \quad \text{and} \quad D^2u(x_0) \leq 0.$$

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$A = (a^{ij}(x_0))_{i,j}$ symmetric & pos. def $\Rightarrow \exists$ orth. matrix O s.t.

$$OAO^{-1} = OAO^T = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} =: D, \quad d_i > 0, \quad i = 1, \dots, n.$$

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$$= \sum_{i=1}^n d_i b_{ii} \leq 0, \quad \text{where } b_{ii} = O_i D^2 u(x_0) O_i^T \leq 0$$

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Hence, $Lu(x_0) = -\sum_{i,j=1}^{n} a^{ij}(x_0) u_{x_i x_j}(x_0) + \sum_{i=1}^{n} b^i(x_0) u_{x_i}(x_0) \geq 0$.

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Contradiction!

2. Assume $Lu \leq 0$ and let $u^{\varepsilon}(x) = u(x) + \varepsilon e^{\lambda x_1}$, $x \in U$, $\varepsilon > 0$.

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$$\sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \geq \theta|\xi|^2 \Rightarrow a^{ii} \geq \theta \quad \forall i \quad \text{(take } \xi = e_i\text{)}.$$

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Hence

$$\begin{aligned} Lu^\varepsilon &= Lu + \varepsilon L(e^{\lambda x_1}) \\ &\le \varepsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) \\ &\le \varepsilon e^{\lambda x_1}(-\lambda^2 \theta + \lambda \|b\|_{L^\infty}) \\ &< 0 \end{aligned}$$

in U if λ is suff. large.

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$$\therefore \max_{\overline{U}} u \leq \max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon \leq \max_{\partial U} u + C\varepsilon.$$

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Let $\varepsilon \downarrow 0$:

$$\max_{\overline{U}} u = \max_{\partial U} u. \qquad \square$$

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If instead $c \geq 0$*, then* $\max_{\overline{U}} u \leq \max_{\partial U} u^+$*.*

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Proof.

Let $V := \{x \in U \colon u(x) > 0\}$ and assume $V \neq \emptyset$ (otherwise obvious).

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$$(L - c)u \leq -cu \leq 0 \quad \text{in } V.$$

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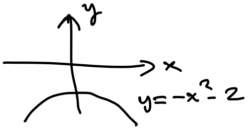
Remark: u^+ cannot be replaced by u as the example $L = -\partial_x^2 + 1$, $u = -x^2 - 2$ shows. $(-\partial_x^2 + 1)u = -x^2 \leq 0$



Application: Uniqueness

Theorem

Under the same assumptions on L, $\exists$ at most one solution $u \in C^2(U) \cap C(\overline{U})$ to the BVP

$$\begin{cases} Lu = f \text{ in } U, \\ \ \ u = g \text{ on } \partial U. \end{cases}$$

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Exercises:

1. Find U violating the interior ball condition.
2. Show that ∂U $C^2 \Rightarrow U$ satisfies int. ball cond. at each $x^0 \in \partial U$.

1\.  or $\{(x,y) : y \geq |x|^{3/2}\}$

(C^1 but not C^2 at $(0,0)$)

2\. Use Taylor expansion.

Lemma (Hopf's lemma)

Assume $u \in C^2(U) \cap C(\overline{U})$ is a subsolution, $c \equiv 0$ in U, and that $\exists x^0 \in \partial U$ s.t. $u(x^0) > u(x)\ \forall x \in U$. Assume also that U satisfies the int. ball. cond. at x^0.

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1. *Then $\frac{\partial u}{\partial \nu}(x^0) > 0$, ν outer unit normal to B (or ∂U) at x^0.*

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Assume $c \geq 0$. Can assume $B = B^0(0, r)$.

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$$Lv = e^{-\lambda |x|^2} \left(\sum_{i,j=1}^{n} a^{ij}(-4\lambda^2 x_i x_j + 2\lambda \delta_{ij}) - \sum_{i=1}^{n} 2b^i \lambda x_i + c \right) - ce^{-\lambda r^2}$$

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&\leq e^{-\lambda |x|^2}(-4\theta \lambda^2 |x|^2 + 2\lambda \| \operatorname{tr} A\|_{L^\infty} + 2\lambda \|b\|_{L^\infty} r + \|c\|_{L^\infty}) \\
&\leq 0
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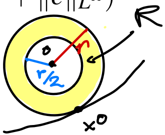
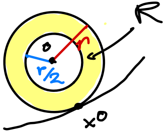


in $R := B \setminus B(0, \frac{r}{2})$ if λ suff. large.

$$u(x^0) > u(x) \quad \forall x \in B$$

$$\Rightarrow \exists \varepsilon > 0 \text{ s.t. } u(x^0) \geq u(x) + \varepsilon v(x) \quad \forall x \in \partial B(0, \tfrac{r}{2})$$



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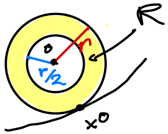
$$\Rightarrow \exists \varepsilon > 0 \text{ s.t. } u(x^0) \geq u(x) + \varepsilon v(x) \quad \forall x \in \partial B(0, \tfrac{r}{2})$$

Also

$$u(x^0) \geq u(x) + \varepsilon v(x) \quad \forall x \in \partial B(0, r)$$

since $v \equiv 0$ there.



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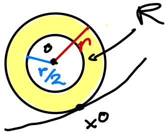

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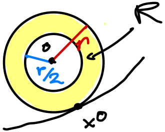

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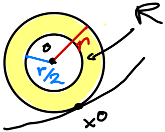

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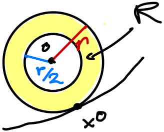

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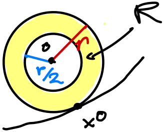

$$\Rightarrow \frac{\partial u}{\partial \nu}(x^0) \geq -\varepsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\varepsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda \varepsilon r e^{-\lambda r^2} > 0. \quad \square$$

Theorem (Strong maximum principle)

Assume $u \in C^2(U) \cap C(\overline{U})$ *and* $c \equiv 0$ *in* U. *Assume also that* U *is connected, open and bounded.*

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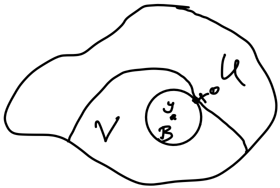

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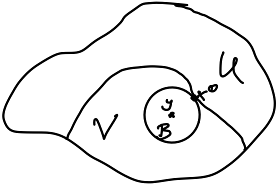

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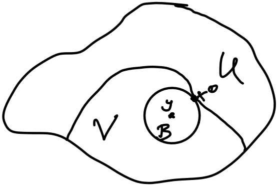

Hopf $\Rightarrow \frac{\partial u}{\partial \nu}(x^0) > 0$.
Contradicts $Du(x^0) = 0$ (x^0 int. max. point). □

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Remark: Nonnegativity is essential as the example $L = -\partial_x^2 + 1$, $u = -x^2 - 2$ shows.