LUND UNIVERSITY

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SOLUTIONS, APRIL 17

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15. Since $P(D)$ is elliptic by assumption, $p_m(\xi) \neq 0$ for all $\xi \neq 0$. But this means that $c := \min_{|\xi|=1} |p_m(\xi)| > 0$. Since p_m is homogeneous of degree m, we get

$$p_m(\xi) = p_m(|\xi|\xi/|\xi|) = |\xi|^m p_m(\xi/|\xi|) \geq c|\xi|^m$$

for $\xi \neq 0$ (and clearly also for $\xi = 0$). Hence,

$$p(\xi) = p_m(\xi) + p(\xi) - p_m(\xi) \geq c|\xi|^m + O(|\xi|^{m-1}) = |\xi|^m(c + O(|\xi|^{-1})) \geq \frac{c}{2}|\xi|^m$$

if $|\xi| \geq R$ for some sufficiently large radius R.
By hypothesis,

$$(1+|\xi|^s)p(\xi)\hat{u} \in L^2(\mathbb{R}^n),$$

and by the above, this means that

$$|\xi|^{s+m}\hat{u} \in L^2(\mathbb{R}^n).$$

Since also $\hat{u} \in L^2(\mathbb{R}^n)$, we get

$$(1+|\xi|^{s+m})\hat{u} \in L^2(\mathbb{R}^n),$$

so that $u \in H^{s+m}(\mathbb{R}^n)$.

16. We have

$$\operatorname{Re} p(\xi) = \operatorname{Re}(a_m)\xi^m.$$

If m is even, this is bounded if and only if $\operatorname{Re}(a_m) \leq 0$. If m is odd, it is bounded if and only if $\operatorname{Re}(a_m) = 0$, that is, if a_m is purely imaginary.

18. Here we only consider $t \geq 0$. Taking the Fourier transform of both sides of the equation

$$u_{tt} = \Delta u,$$

which holds as an equality between H^s functions, we get

$$\hat{u}_{tt}(\xi,t) = -|\xi|^2\hat{u}(\xi,t) \tag{1}$$

for almost every ξ and $\hat{u}(\xi,0) = \hat{g}(\xi)$, $\hat{u}_t(\xi,0) = \hat{h}(\xi)$. Clearly one solution is

$$\hat{u}(\xi,t) = \cos(|\xi|t)\hat{g}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{h}(\xi).$$

Please, turn over!

Noting that

$$\frac{|\sin(|\xi|t)|}{|\xi|} \le \min\{t, |\xi|^{-1}\} \le \frac{1+t}{1+|\xi|},$$

we can estimate

$$|\hat{u}(\xi,t)| \le |\hat{g}(\xi)| + \frac{1+t}{1+|\xi|}|\hat{h}(\xi)|.$$

Therefore, for each fixed t (or uniformly over any interval $[0,T]$, $T>0$), we can estimate the H^{s+2} norm of u by the H^{s+2} norm of g and the H^{s+1} norm of h (recall that $(1+|\xi|^{s+2})/(1+|\xi|) \le (1+|\xi|^{s+1})$). Similarly,

$$\hat{u}_t(\xi,t) = -|\xi|\sin(|\xi|t)\hat{g}(\xi) + \cos(|\xi|t)\hat{h}(\xi).$$

giving

$$|\hat{u}_t(\xi,t)| \le |\xi||\hat{g}(\xi)| + |\hat{h}(\xi)|.$$

From this it is clear that the H^{s+1} norm of u_t can be estimated in the same way.
To see that the solution is continuous as a function of t with values in $H^{s+2}(\mathbb{R}^n)$ one can use the dominated convergence theorem. Note that

$$\begin{aligned}\|u(\cdot,t+\tau)-u(\cdot,t)\|_{H^{s+2}} \le &\left(\int_{\mathbb{R}^n}(1+|\xi|^{s+2})^2(\cos(|\xi|(t+\tau))-\cos(|\xi|t))^2|\hat{g}(\xi)|^2\,d\xi\right)^{1/2}\\ &+\left(\int_{\mathbb{R}^n}(1+|\xi|^{s+2})^2\frac{(\sin(|\xi|(t+\tau))-\sin(|\xi|t))^2}{|\xi|^2}|\hat{h}(\xi)|^2\,d\xi\right)^{1/2}\end{aligned}$$

with both integrands tending pointwise to 0 as $\tau \to 0$, and with

$$(1+|\xi|^{s+2})(\cos(|\xi|(t+\tau))-\cos(|\xi|t))^2|\hat{g}(\xi)|^2 \le 4(1+|\xi|^{s+2})|\hat{g}(\xi)|^2 \in L^1$$

and

$$\begin{aligned}(1+|\xi|^{s+2})^2\frac{(\sin(|\xi|(t+\tau))-\sin(|\xi|t))^2}{|\xi|^2}|\hat{h}(\xi)|^2 &\le \frac{(1+|\xi|^{s+2})^2}{(1+|\xi|)^2}(2+2t+\tau)^2|\hat{h}(\xi)|^2\\ &\le (1+|\xi|^{s+1})^2(3+2t)^2|\hat{h}(\xi)|^2 \in L^1\end{aligned}$$

if $|\tau| \le 1$. To show that $u \in C^1([0,\infty); H^{s+1}(\mathbb{R}^n))$ one can mimic the proof of Theorem 4.25, using the inequalities

$$|\cos(x+y)-\cos(x)| \le |y| \quad \text{and} \quad |\sin(x+y)-\sin(x)| \le |y|.$$

Repeating this gives $u \in C^2([0,\infty); H^s(\mathbb{R}^n))$.
For the uniqueness, one possibility is to use the energy method. Assuming that u is a solution in $C([0,T); H^2(\mathbb{R}^n)) \cap C^1([0,T); H^1(\mathbb{R}^n)) \cap C^2([0,T); L^2(\mathbb{R}^n))$ (but without assuming that it is given by the above formula), we get

$$\int_{\mathbb{R}^n}(u_t^2+|Du|^2)\,dx = \int_{\mathbb{R}^n}(|\hat{u}_t|^2+|\xi|^2|\hat{u}|^2)\,d\xi$$

and therefore

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}(u_t^2+|Du|^2)\,dx &= 2\,\mathrm{Re}\int_{\mathbb{R}^n}(\hat{u}_t\bar{\hat{u}}_{tt}+|\xi|^2\hat{u}\bar{\hat{u}}_t)\,dx\\ &= 2\,\mathrm{Re}\int_{\mathbb{R}^n}(u_{tt}+|\xi|^2\hat{u})\bar{\hat{u}}_t\,dx\\ &= 0\end{aligned}$$

by (1). Thus if $g=h=0$, we get $u_t=0$. But this implies that $u=g=0$ for all t by Proposition 4.23.
Alternatively, one can use uniqueness for ODEs, but it's a little bit tricky since there is an 'almost everywhere' issue in (1).

Evans 5.10

1. For simplicity we consider only the case $k=0$. The case $k>0$ is similar (recall that if $u_k \in C^1$ and $u_k \to u$, $Du_k \to v$ uniformly, then u is C^1 with $Du = v$). The fact that $C^{0,\gamma}(\overline{U})$ is a normed vector space is easy to show, so we concentrate on showing completeness. We already know that $C(\overline{U})$ is complete, so it suffices to show that if $\{u_n\}$ is a Cauchy sequence in $C^{0,\gamma}(\overline{U})$, and $u \in C(\overline{U})$, then $u \in C^{0,\gamma}(\overline{U})$ and $[u_n - u]_{C^{0,\gamma}(\overline{U})} \to 0$. By the definition of a Cauchy sequence, we can for each $\varepsilon > 0$ find an N such that

$$\frac{|u_n(x) - u_n(y) - (u_m(x) - u_m(y))|}{|x-y|^\gamma} \le \varepsilon, \qquad x, y \in U, \quad x \neq y$$

if $n, m \ge N$. Hence,

$$\frac{|u_n(x) - u_n(y)|}{|x-y|^\gamma} \le C, \qquad x, y \in U, \quad x \neq y$$

for all $n \ge N$, where $C := \varepsilon + [u_N]_{C^{0,\gamma}}$. Letting $n \to \infty$, we get

$$\frac{|u(x) - u(y)|}{|x-y|^\gamma} \le C, \qquad x, y \in U, \quad x \neq y.$$

Thus $u \in C^{0,\gamma}(\overline{U})$. On the other hand, letting $m \to \infty$ above we get

$$\frac{|u_n(x) - u_n(y) - (u(x) - u(y))|}{|x-y|^\gamma} \le \varepsilon, \qquad x, y \in U, \quad x \neq y$$

for $n \ge N$, which implies that $[u_n - u]_{C^{0,\gamma}} \to 0$.

20. We have

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi.$$

and thus

$$\begin{aligned} |u(x)| &\le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\hat{u}(\xi)|\, d\xi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{1+|\xi|^s}{1+|\xi|^s} |\hat{u}(\xi)|\, d\xi \\ &\le \frac{1}{(2\pi)^{n/2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^s)^2}\, d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^s)^2 |\hat{u}(\xi)|^2\, d\xi \right)^{1/2}. \end{aligned}$$

The first integral converges if $s > n/2$ and then the result holds with

$$C = \frac{1}{(2\pi)^{n/2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^s)^2}\, d\xi \right)^{1/2}.$$

Evans 6.6

2. We already know that there is a constant $\alpha > 0$ such that

$$|B[u,v]| \le \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

from Theorem 2. It remains to show the lower bound. We have

$$B[u,u] = \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} + cu^2 \right) dx \ge \theta \|Du\|^2_{L^2(U)} - \mu \|u\|^2_{L^2(U)}.$$

Please, turn over!

From Poincaré's inequality, we know that there are constants $C_1, C_2 > 0$ such that

$$\|u\|_{L^2(U)} \le C_1 \|Du\|_{L^2(U)}.$$

and

$$\|u\|_{H_0^1(U)} \le C_2 \|Du\|_{L^2(U)}$$

for $u \in H_0^1(U)$. Thus,

$$B[u,u] \ge (\theta - C_1^2\mu)\|Du\|_{L^2(U)}^2 \ge (\theta - C_1^2\mu)C_2^2\|u\|_{H_0^1(U)}^2.$$

The hypotheses of the Lax-Milgram theorem are therefore satisfied with $\beta = (\theta - C_1^2\mu)C_2^2$ if $\mu < \theta/C_1^2$.

3. We verify the hypothesis of the Lax-Milgram theorem (Riesz' representation theorem can also be used) with

$$B[u,v] = \int_U \Delta u \Delta v\, dx, \quad u,v \in H_0^2(U).$$

Clearly there is a constant $C > 0$ such that

$$|B[u,v]| \le C\|D^2u\|_{L^2(U)}\|D^2v\|_{L^2(U)} \le C\|u\|_{H_0^2(U)}\|v\|_{H_0^2(U)}.$$

To prove the lower bound, note that

$$B[u,u] = \int_U (\Delta u)^2\, dx.$$

If $u \in C_c^\infty(U)$, then integration by parts gives

$$\int_U (\Delta u)^2\, dx = \int_U \sum_{i,j=1}^n u_{x_ix_i}u_{x_jx_j}\, dx = \sum_{i,j=1}^n \int_U u_{x_ix_j}^2\, dx$$

and by approximation this still holds if $u \in H_0^2(U)$. On the other hand,

$$\int_U u_{x_i}^2\, dx \le C\sum_{j=1}^n \int_U u_{x_ix_j}^2\, dx$$

by Poincaré's inequality if $u_{x_i} \in H_0^1(U)$, so

$$\sum_{i=1}^n \int_U u_{x_i}^2\, dx \le C\sum_{i,j=1}^n \int_U u_{x_ix_j}^2\, dx$$

if $u \in H_0^2(U)$. Finally,

$$\|u\|_{L^2(U)}^2 \le C\|Du\|_{L^2(U)}^2 \le C^2\sum_{i,j=1}^n \int_U u_{x_ix_j}^2\, dx$$

if $u \in H_0^2(U)$. Altogether, we have proved that

$$\beta\|u\|_{H_0^2(U)}^2 \le \sum_{i,j=1}^n \int_U u_{x_ix_j}^2\, dx = B[u,u], \quad u \in H_0^2(U),$$

where $\beta = 1/(1+C+C^2)$.

4. For the ‘only if’ part, we note that

$$\int_U f\,dx = \int_U f \cdot 1\,dx = \int_U Du \cdot D1\,dx = 0$$

if u is a weak solution. To show the existence of a weak solution, assuming the necessary condition, we apply Lax-Milgram to the bilinear form

$$B[u,v] = \int_U Du \cdot Dv\,dx$$

on the space

$$H = \left\{ u \in H^1(U) : \int_U u\,dx = 0 \right\}.$$

The upper bound is clear, so we concentrate on the lower bound. Here we need Theorem 1 in Evans, 5.8, which says that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(U)} \leq C\|Du\|_{L^2(U)}$$

for each $u \in H$. This implies that the ‘homogeneous Sobolev norm’ $\|Du\|_{L^2(U)}$ is equivalent to the usual Sobolev norm $\|u\|_{H^1(U)}$ on H and hence there is a constant $\beta > 0$ with

$$\beta\|u\|^2_{H^1(U)} \leq \|Du\|^2_{L^2(U)} = B[u,u], \quad u \in H.$$

Lax-Milgram gives the existence of a $u \in H$ s.t. $B[u,v] = (f,v)_{L^2(U)}$ for all $v \in H$. We are still not completely done, since we want $B[u,v] = (f,v)_{L^2(U)}$ for all $v \in H^1(U)$. To get this, note that if $v \in H^1(U)$, then $v - (v)_U \in H$, where $(v)_U := |U|^{-1}\int_U v\,dx$ is the average of v over U. But then

$$B[u,v] = B[u,v-(v)_U] + B[u,(v)_U] = B[u,v-(v)_U] = (f,v-(v)_U)_{L^2(U)} = (f,v)_{L^2(U)}$$

since $D((v)_U) = 0$ and $(f,(v)_U)_{L^2(U)}$.