PDE Lecture

Sobolev spaces

April 14

Fourier methods and Sobolev spaces

Fourier characterization of Sobolev spaces

Recall:
Let $u \in L^2(\mathbb{R}^n)$.
$u \in H^k(\mathbb{R}^n) \Leftrightarrow (1+|\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

Fourier characterization of Sobolev spaces

Recall:
Let $u \in L^2(\mathbb{R}^n)$.
$u \in H^k(\mathbb{R}^n) \Leftrightarrow (1+|\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

Fractional Sobolev norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|\xi|^s)\hat{u}\|_{L^2} = \left(\int_{\mathbb{R}^n} (1+|\xi|^s)^2 |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2}, \quad s \geq 0.$$

Fourier characterization of Sobolev spaces

Recall:
Let $u \in L^2(\mathbb{R}^n)$.
$u \in H^k(\mathbb{R}^n) \Leftrightarrow (1+|\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

Fractional Sobolev norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|\xi|^s)\hat{u}\|_{L^2} = \left(\int_{\mathbb{R}^n} (1+|\xi|^s)^2 |\hat{u}(\xi)|^2 \, d\xi\right)^{1/2}, \quad s \geq 0.$$

Definition

Let $s \geq 0$. The fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \colon \|u\|_{H^s} < \infty\}.$$

Fourier characterization of Sobolev spaces

Recall:
Let $u \in L^2(\mathbb{R}^n)$.
$u \in H^k(\mathbb{R}^n) \Leftrightarrow (1+|\xi|^k)\hat{u} \in L^2(\mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

Fractional Sobolev norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1+|\xi|^s)\hat{u}\|_{L^2} = \left(\int_{\mathbb{R}^n} (1+|\xi|^s)^2 |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}, \quad s \geq 0.$$

Definition

Let $s \geq 0$. The fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \colon \|u\|_{H^s} < \infty\}.$$

Coincides with H^k if $s = k = 0, 1, 2, \ldots$

Application to PDE with constant coefficients

Example

Consider

$$-\Delta u + u = f$$

with $u, f \in \mathscr{S}'(\mathbb{R}^n)$.

Application to PDE with constant coefficients

Example

Consider

$$-\Delta u + u = f$$

with $u, f \in \mathscr{S}'(\mathbb{R}^n)$.

Fourier transform

$$(|\xi|^2 + 1)\hat{u} = \hat{f} \Leftrightarrow \hat{u} = \frac{1}{1+|\xi|^2}\hat{f}$$

Application to PDE with constant coefficients

Example

Consider

$$-\Delta u + u = f$$

with $u,f \in \mathscr{S}'(\mathbb{R}^n)$.

Fourier transform

$$(|\xi|^2 + 1)\hat{u} = \hat{f} \Leftrightarrow \hat{u} = \frac{1}{1+|\xi|^2}\hat{f}$$

Thus $f \in L^2 \Rightarrow u \in H^2$.

Application to PDE with constant coefficients

Example

Consider

$$-\Delta u + u = f$$

with $u, f \in \mathscr{S}'(\mathbb{R}^n)$.

Fourier transform

$$(|\xi|^2 + 1)\hat{u} = \hat{f} \Leftrightarrow \hat{u} = \frac{1}{1+|\xi|^2}\hat{f}$$

Thus $f \in L^2 \Rightarrow u \in H^2$.

$f \in H^s \Rightarrow u \in H^{s+2}$ since

$$\frac{1+|\xi|^{s+2}}{1+|\xi|^2} \leq 1 + |\xi|^s$$

Elliptic equations with constant coefficients

General partial differential operator of order m

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha.$$

Elliptic equations with constant coefficients

General partial differential operator of order m

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha.$$

Symbol:

$$p(\xi) = \sum_{|\alpha| \le m} a_\alpha (i\xi)^\alpha$$

Elliptic equations with constant coefficients

General partial differential operator of order m

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha.$$

Symbol:

$$p(\xi) = \sum_{|\alpha| \leq m} a_\alpha (i\xi)^\alpha$$

Principal symbol:

$$p_m(\xi) = \sum_{|\alpha| = m} a_\alpha (i\xi)^\alpha.$$

Elliptic equations with constant coefficients

General partial differential operator of order m

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha.$$

Symbol:

$$p(\xi) = \sum_{|\alpha| \leq m} a_\alpha (i\xi)^\alpha$$

Principal symbol:

$$p_m(\xi) = \sum_{|\alpha| = m} a_\alpha (i\xi)^\alpha.$$

$P(D)$ is <u>elliptic</u> if $p_m(\xi) \neq 0 \; \forall \; \xi \neq 0$.

Theorem

Assume that $P(D)$ is elliptic and $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. If $f \in H^s$, $s \geq 0$, then $P(D)u = f$ has a unique solution $u \in H^{s+m}$ and the map $f \mapsto u$ from H^s to H^{s+m} is continuous.

Theorem

Assume that $P(D)$ is elliptic and $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. If $f \in H^s$, $s \geq 0$, then $P(D)u = f$ has a unique solution $u \in H^{s+m}$ and the map $f \mapsto u$ from H^s to H^{s+m} is continuous.

Proof.

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{p(\xi)}$$

□

Theorem

Assume that $P(D)$ is elliptic and $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. If $f \in H^s$, $s \geq 0$, then $P(D)u = f$ has a unique solution $u \in H^{s+m}$ and the map $f \mapsto u$ from H^s to H^{s+m} is continuous.

Proof.

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{p(\xi)}$$

Assumptions $\Rightarrow |p(\xi)| \geq c(1 + |\xi|^m)$ for some $c > 0$ (exercise).

$\square$

Theorem

Assume that $P(D)$ *is elliptic and* $p(\xi) \neq 0$ *for all* $\xi \in \mathbb{R}^n$*. If* $f \in H^s$*,* $s \geq 0$*, then* $P(D)u = f$ *has a unique solution* $u \in H^{s+m}$ *and the map* $f \mapsto u$ *from* H^s *to* H^{s+m} *is continuous.*

Proof.

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{p(\xi)}$$

Assumptions $\Rightarrow |p(\xi)| \geq c(1 + |\xi|^m)$ for some $c > 0$ (exercise).

$$(1 + |\xi|^{s+m})|\hat{u}(\xi)| \leq \frac{1 + |\xi|^{s+m}}{c(1 + |\xi|^m)}|\hat{f}(\xi)|$$

$\square$

Theorem

Assume that $P(D)$ is elliptic and $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. If $f \in H^s$, $s \geq 0$, then $P(D)u = f$ has a unique solution $u \in H^{s+m}$ and the map $f \mapsto u$ from H^s to H^{s+m} is continuous.

Proof.

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{p(\xi)}$$

Assumptions $\Rightarrow |p(\xi)| \geq c(1+|\xi|^m)$ for some $c > 0$ (exercise).

$$\begin{aligned}(1+|\xi|^{s+m})|\hat{u}(\xi)| &\leq \frac{1+|\xi|^{s+m}}{c(1+|\xi|^m)}|\hat{f}(\xi)| \\ &\leq \frac{1+|\xi|^s}{c}|\hat{f}(\xi)|\end{aligned}$$

□

Theorem

Assume that $P(D)$ is elliptic and $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. If $f \in H^s$, $s \geq 0$, then $P(D)u = f$ has a unique solution $u \in H^{s+m}$ and the map $f \mapsto u$ from H^s to H^{s+m} is continuous.

Proof.

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{p(\xi)}$$

Assumptions $\Rightarrow |p(\xi)| \geq c(1 + |\xi|^m)$ for some $c > 0$ (exercise).

$$\begin{aligned}(1 + |\xi|^{s+m})|\hat{u}(\xi)| &\leq \frac{1 + |\xi|^{s+m}}{c(1 + |\xi|^m)}|\hat{f}(\xi)| \\ &\leq \frac{1 + |\xi|^s}{c}|\hat{f}(\xi)|\end{aligned}$$

$$\Rightarrow \|u\|_{H^{s+m}} \leq \frac{1}{c}\|f\|_{H^s}$$

□

This implies that the problem is well-posed:

This implies that the problem is well-posed:

1. the problem has a solution;

This implies that the problem is well-posed:

1. the problem has a solution;
2. the solution is unique;

This implies that the problem is well-posed:

1. the problem has a solution;
2. the solution is unique;
3. the solution depends continuously on the data.

This implies that the problem is well-posed:

1. the problem has a solution;
2. the solution is unique;
3. the solution depends continuously on the data.

Q1: How do you get the estimate $|p(\xi)| \geq c(1+|\xi|^m)$?

Q2: Can you think of an operator such that $p(\xi) \neq 0$ for all ξ, but $P(D)$ is not elliptic?

Evolution equations

Example

$$\begin{cases} u_t = \Delta u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Evolution equations

Example

$$\begin{cases} u_t = \Delta u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Assume $g \in H^{s+2}(\mathbb{R}^n)$.

Evolution equations

Example

$$\begin{cases} u_t = \Delta u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Assume $g \in H^{s+2}(\mathbb{R}^n)$.

Look at the solution as map $u\colon [0,\infty) \to H^{s+2}(\mathbb{R}^n)$.

Evolution equations

Example

$$\begin{cases} u_t = \Delta u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Assume $g \in H^{s+2}(\mathbb{R}^n)$.

Look at the solution as map $u\colon [0,\infty) \to H^{s+2}(\mathbb{R}^n)$.

$$\hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{g}(\xi) \Rightarrow |\hat{u}(\xi,t)| \leq |\hat{g}(\xi)|$$

Evolution equations

Example

$$\begin{cases} u_t = \Delta u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Assume $g \in H^{s+2}(\mathbb{R}^n)$.

Look at the solution as map $u\colon [0,\infty) \to H^{s+2}(\mathbb{R}^n)$.

$$\begin{aligned} \hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{g}(\xi) &\Rightarrow |\hat{u}(\xi,t)| \le |\hat{g}(\xi)| \\ &\Rightarrow u(\cdot,t) \in H^{s+2}(\mathbb{R}^n), \quad t \ge 0. \end{aligned}$$

Evolution equations

Example

$$\begin{cases} u_t = \Delta u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Assume $g \in H^{s+2}(\mathbb{R}^n)$.

Look at the solution as map $u\colon [0,\infty) \to H^{s+2}(\mathbb{R}^n)$.

$$\begin{aligned} \hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{g}(\xi) &\Rightarrow |\hat{u}(\xi,t)| \le |\hat{g}(\xi)| \\ &\Rightarrow u(\cdot,t) \in H^{s+2}(\mathbb{R}^n), \quad t \ge 0. \end{aligned}$$

Dominated convergence $\Rightarrow u \in C([0,T);H^{s+2}(\mathbb{R}^n))$.

Similarly $u \in C^1([0,T);H^s(\mathbb{R}^n))$:

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-u_t(t)\right\|_{H^s}=0$$

and $u_t=\Delta u \in C([0,T);H^s(\mathbb{R}^n))$.

Similarly $u \in C^1([0,T);H^s(\mathbb{R}^n))$:

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-u_t(t)\right\|_{H^s}=0$$

and $u_t = \Delta u \in C([0,T);H^s(\mathbb{R}^n))$.

More generally, consider

$$\begin{cases} u_t = P(D)u, & t>0, \\ u=g, & t=0. \end{cases} \tag{1}$$

Similarly $u \in C^1([0,T);H^s(\mathbb{R}^n))$:

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h} - u_t(t)\right\|_{H^s} = 0$$

and $u_t = \Delta u \in C([0,T);H^s(\mathbb{R}^n))$.

More generally, consider

(1)
$$\begin{cases} u_t = P(D)u, & t > 0, \\ u = g, & t = 0. \end{cases}$$

Theorem

Assume that $\alpha := \sup_{\xi\in\mathbb{R}^n} \operatorname{Re} p(\xi) < \infty$*. Then* (1) *has a unique solution* $u \in C([0,\infty);H^{s+m}(\mathbb{R}^n)) \cap C^1([0,\infty);H^s(\mathbb{R}^n))$ *for each* $g \in H^{s+m}(\mathbb{R}^n)$*,* $s \geq 0$*. The solution satisfies*

$$\|u(\cdot,t)\|_{H^{s+m}(\mathbb{R}^n)} \leq e^{\alpha t}\|g\|_{H^{s+m}(\mathbb{R}^n)}$$

Similarly $u \in C^1([0,T);H^s(\mathbb{R}^n))$:

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-u_t(t)\right\|_{H^s}=0$$

and $u_t = \Delta u \in C([0,T);H^s(\mathbb{R}^n))$.

More generally, consider

(1) $$\begin{cases} u_t = P(D)u, & t>0, \\ u = g, & t=0. \end{cases}$$

Theorem

Assume that $\alpha := \sup_{\xi\in\mathbb{R}^n} \operatorname{Re} p(\xi) < \infty$. Then (1) *has a unique solution $u \in C([0,\infty);H^{s+m}(\mathbb{R}^n)) \cap C^1([0,\infty);H^s(\mathbb{R}^n))$ for each $g \in H^{s+m}(\mathbb{R}^n)$, $s \geq 0$. The solution satisfies*

$$\|u(\cdot,t)\|_{H^{s+m}(\mathbb{R}^n)} \leq e^{\alpha t}\|g\|_{H^{s+m}(\mathbb{R}^n)}$$

Proof: see Wahlén, Theorem 4.25.

2nd order elliptic equations with variable coefficients

(2) $$\begin{cases} Lu = f \text{ in } U, \\ \quad u = 0 \text{ on } \partial U \end{cases}$$

$$(2) \qquad \begin{cases} Lu = f \text{ in } U, \\ u = 0 \text{ on } \partial U \end{cases}$$

Divergence form:

$$(3) \qquad Lu = -\sum_{ij} (a^{ij}(x) u_{x_i})_{x_j} + \sum_i b^i(x) u_{x_i} + c(x) u$$

(2)
$$\begin{cases} Lu = f \text{ in } U, \\ \quad u = 0 \text{ on } \partial U \end{cases}$$

Divergence form:

(3)
$$Lu = -\sum_{ij} (a^{ij}(x) u_{x_i})_{x_j} + \sum_i b^i(x) u_{x_i} + c(x) u$$

Non-divergence form:

(4)
$$Lu = -\sum_{ij} a^{ij}(x) u_{x_i x_j} + \sum_i b^i(x) u_{x_i} + c(x) u$$

$$
(2) \qquad \begin{cases} Lu = f \text{ in } U, \\ u = 0 \text{ on } \partial U \end{cases}
$$

Divergence form:

$$
(3) \qquad Lu = -\sum_{ij} (a^{ij}(x) u_{x_i})_{x_j} + \sum_i b^i(x) u_{x_i} + c(x) u
$$

Non-divergence form:

$$
(4) \qquad Lu = -\sum_{ij} a^{ij}(x) u_{x_i x_j} + \sum_i b^i(x) u_{x_i} + c(x) u
$$

Remark

If $a^{ij} \in C^1$ we can rewrite (3) in the form (4) with

$$
\tilde{b}^i = b^i - \sum_{j=1}^{n} a^{ij}_{x_j}
$$

(and vice versa).

We assume $a^{ij} = a^{ji}$ (OK if $a^{ij} \in C^1$ by modifying b^i)

We assume $a^{ij} = a^{ji}$ (OK if $a^{ij} \in C^1$ by modifying b^i)

Definition

L is uniformly elliptic if $\exists \theta > 0$ s.t.

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall x \in U \text{ and } \xi \in \mathbb{R}^n.$$

We assume $a^{ij} = a^{ji}$ (OK if $a^{ij} \in C^1$ by modifying b^i)

Definition

L is <u>uniformly elliptic</u> if $\exists \theta > 0$ s.t.

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall x \in U \text{ and } \xi \in \mathbb{R}^n.$$

Example: $L = -\Delta$, $a^{ij} = \delta_{ij}$, $\theta = 1$.

Weak solutions

We consider L in divergence form, a^{ij}, b^i, $c \in L^\infty(U), f \in L^2(U)$.

Weak solutions

We consider L in divergence form, a^{ij}, b^i, $c \in L^\infty(U)$, $f \in L^2(U)$.
Assume u C^2 solution. Multiply $Lu = f$ by $v \in C_c^\infty(U)$ and integrate:

$$B[u,v] := \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx = \int_U fv \, dx$$

Weak solutions

We consider L in divergence form, a^{ij}, b^i, $c \in L^\infty(U)$, $f \in L^2(U)$.

Assume u C^2 solution. Multiply $Lu = f$ by $v \in C_c^\infty(U)$ and integrate:

$$B[u,v] := \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx = \int_U fv\, dx$$

Weak solutions

We consider L in divergence form, a^{ij}, b^i, $c \in L^\infty(U)$, $f \in L^2(U)$.

Assume u C^2 solution. Multiply $Lu = f$ by $v \in C_c^\infty(U)$ and integrate:

$$B[u,v] := \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx = \int_U fv \, dx$$

Continues to hold if $v \in H_0^1$ and still makes sense if $u \in H_0^1$.

Weak solutions

We consider L in divergence form, a^{ij}, b^i, $c \in L^\infty(U)$, $f \in L^2(U)$.
Assume u C^2 solution. Multiply $Lu = f$ by $v \in C_c^\infty(U)$ and integrate:

$$B[u,v] := \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx = \int_U fv \, dx$$

Continues to hold if $v \in H_0^1$ and still makes sense if $u \in H_0^1$.

Definition

$u \in H_0^1(U)$ is a weak solution of (2) if

$$B[u,v] = (f,v)_{L^2(U)} \quad \forall v \in H_0^1(U).$$

Weak solutions

We consider L in divergence form, a^{ij}, b^i, $c \in L^\infty(U)$, $f \in L^2(U)$.
Assume u C^2 solution. Multiply $Lu = f$ by $v \in C_c^\infty(U)$ and integrate:

$$B[u,v] := \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx = \int_U fv\, dx$$

Continues to hold if $v \in H_0^1$ and still makes sense if $u \in H_0^1$.

Definition

$u \in H_0^1(U)$ is a weak solution of (2) if

$$B[u,v] = (f,v)_{L^2(U)} \quad \forall v \in H_0^1(U).$$

Remark: Evans also discusses solutions for $f \in H^{-1}(U)$.

A bit of functional analysis

Evans D.2–D.3, 6.2.1

H real Hilbert space.

A bit of functional analysis

Evans D.2–D.3, 6.2.1
H real Hilbert space.

Definition

A <u>bounded linear functional</u> on H is a linear operator $u^*\colon H \to \mathbb{R}$ s.t. $\exists\ C \geq 0$ with

$$|u^*(v)| \leq C\|v\| \quad \forall v \in H.$$

We also write $\langle u^*, v\rangle := u^*(v)$.
The <u>dual space</u> H^* is the space of bdd linear functionals on H.

A bit of functional analysis

Evans D.2–D.3, 6.2.1
H real Hilbert space.

Definition

A bounded linear functional on H is a linear operator $u^*\colon H \to \mathbb{R}$ s.t. $\exists\ C \geq 0$ with

$$|u^*(v)| \leq C\|v\| \quad \forall v \in H.$$

We also write $\langle u^*, v\rangle := u^*(v)$.
The dual space H^* is the space of bdd linear functionals on H.

Example:
Let $u \in H$. Then $u^*\colon v \mapsto (u,v)_H$ is a bdd linear functional with

$$|u^*(v)| \leq \|u\|\|v\|.$$

Theorem (Riesz' representation theorem)

H^* can be identified with H in a canonical way: $\forall u^* \in H^*$ $\exists$ unique $u \in H$ s.t.

$$\langle u^*, v\rangle = (u, v)_H \quad \forall v \in H.$$

The mapping $u^* \mapsto u$ is a linear isomorphism of H^* onto H.

Theorem (Riesz' representation theorem)

H^* can be identified with H in a canonical way: $\forall u^* \in H^*$ $\exists$ unique $u \in H$ s.t.

$$\langle u^*, v\rangle = (u,v)_H \quad \forall v \in H.$$

The mapping $u^* \mapsto u$ is a linear isomorphism of H^* onto H.

Proof: Functional analysis course.

Theorem (Lax-Milgram)

Assume $B\colon H \times H \to \mathbb{R}$ *is a bilinear mapping s.t.* $\exists \alpha, \beta > 0$ *with*

1. $|B[u,v]| \leq \alpha \|u\| \|v\| \ \forall u, v \in H$ *(boundedness)*
2. $\beta \|u\|^2 \leq B[u,u]$ *(coercivity)*

and let $f \in H^*$*.*

Theorem (Lax-Milgram)

Assume $B\colon H\times H\to\mathbb{R}$ *is a bilinear mapping s.t.* $\exists\alpha,\beta>0$ *with*

1. $|B[u,v]|\leq\alpha\|u\|\|v\|\ \forall u,v\in H$ *(boundedness)*
2. $\beta\|u\|^2\leq B[u,u]$ *(coercivity)*

and let $f\in H^*$*.*
Then $\exists$ *unique* $u\in H$ *s.t.*

$$B[u,v]=\langle f,v\rangle\quad\forall v\in H.$$

Theorem (Lax-Milgram)

Assume $B\colon H\times H\to\mathbb{R}$ *is a bilinear mapping s.t.* $\exists\alpha,\beta>0$ *with*

1. $|B[u,v]|\le\alpha\|u\|\|v\|\ \forall u,v\in H$ *(boundedness)*
2. $\beta\|u\|^2\le B[u,u]$ *(coercivity)*

and let $f\in H^*$*.*
Then $\exists$ *unique* $u\in H$ *s.t.*

$$B[u,v]=\langle f,v\rangle\quad\forall v\in H.$$

Remark

- $(\cdot,\cdot)_H$ is a bilinear form satisfying the assumptions, so Riesz' representation thm is a special case.

Theorem (Lax-Milgram)

Assume $B\colon H \times H \to \mathbb{R}$ *is a bilinear mapping s.t.* $\exists \alpha, \beta > 0$ *with*

1. $|B[u,v]| \leq \alpha \|u\| \|v\| \; \forall u, v \in H$ *(boundedness)*
2. $\beta \|u\|^2 \leq B[u,u]$ *(coercivity)*

and let $f \in H^*$*.*
Then $\exists$ *unique* $u \in H$ *s.t.*

$$B[u,v] = \langle f, v \rangle \quad \forall v \in H.$$

Remark

- $(\cdot,\cdot)_H$ is a bilinear form satisfying the assumptions, so Riesz' representation thm is a special case.
- If B is symmetric, $B[u,v] = B[v,u]$, then Lax-Milgram follows from Riesz', by using this as an inner product on H.

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \; \forall v \in H$.

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

2. A is bdd:

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

2. A is bdd:

$$\|Au\|^2 = (Au, Au)_H = B[u, Au] \leq \alpha \|u\| \|Au\|$$

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

2. A is bdd:

$$\begin{aligned} &\|Au\|^2 = (Au, Au)_H = B[u, Au] \le \alpha \|u\| \|Au\| \\ \Rightarrow &\|Au\| \le \alpha \|u\|. \end{aligned}$$

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A: H \to H$ linear operator.

2. A is bdd:

$$
\begin{aligned}
\|Au\|^2 &= (Au, Au)_H = B[u, Au] \leq \alpha \|u\| \|Au\| \\
\Rightarrow &\|Au\| \leq \alpha \|u\|.
\end{aligned}
$$

3. A is injective. Indeed

(*)

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

2. A is bdd:

$$\begin{aligned} &\|Au\|^2 = (Au,Au)_H = B[u,Au] \le \alpha\|u\|\|Au\| \\ \Rightarrow &\|Au\| \le \alpha\|u\|. \end{aligned}$$

3. A is injective. Indeed

$$\beta\|u\|^2 \le B[u,u] = (Au,u)_H \le \|Au\|\|u\|$$

(*)

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \; \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

2. A is bdd:

$$\begin{aligned} &\|Au\|^2 = (Au, Au)_H = B[u, Au] \leq \alpha \|u\| \|Au\| \\ \Rightarrow &\|Au\| \leq \alpha \|u\|. \end{aligned}$$

3. A is injective. Indeed

$$\begin{aligned} &\beta \|u\|^2 \leq B[u,u] = (Au, u)_H \leq \|Au\| \|u\| \\ \text{(*)} \qquad &\Rightarrow \beta \|u\| \leq \|Au\|. \end{aligned}$$

Proof.

1. $\forall u \in H$, $v \mapsto B[u,v]$ is a bdd, lin. functional. on H.
Riesz $\Rightarrow \exists w \in H$ s.t. $B[u,v] = (w,v)_H \ \forall v \in H$.
Write $w = Au$, $A\colon H \to H$ linear operator.

2. A is bdd:

$$\begin{aligned} \|Au\|^2 &= (Au,Au)_H = B[u,Au] \le \alpha\|u\|\|Au\| \\ \Rightarrow &\|Au\| \le \alpha\|u\|. \end{aligned}$$

3. A is injective. Indeed

$$\begin{aligned} \beta\|u\|^2 &\le B[u,u] = (Au,u)_H \le \|Au\|\|u\| \\ \text{(*)} \qquad \Rightarrow &\beta\|u\| \le \|Au\|. \end{aligned}$$

$Au = 0 \Rightarrow u = 0$, so A is injective.

4. A has closed range. Indeed

4. A has closed range. Indeed

$$Au_k \to v \Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty$$

4. A has closed range. Indeed

$$
\begin{aligned}
Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty \\
&\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty \qquad (*)
\end{aligned}
$$

4. A has closed range. Indeed

$$\begin{aligned} Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty \\ &\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty && (*) \\ &\Rightarrow u_k \to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \end{aligned}$$

4. A has closed range. Indeed

$$
\begin{aligned}
Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty \\
&\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty && (*) \\
&\Rightarrow u_k \to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\
&\Rightarrow v = Au && (A \text{ continuous})
\end{aligned}
$$

4. A has closed range. Indeed

$$\begin{aligned}
Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty && \\
&\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty && (*) \\
&\Rightarrow u_k \to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\
&\Rightarrow v = Au && (A \text{ continuous})
\end{aligned}$$

5. $R(A) = H$:

4. A has closed range. Indeed

$$
\begin{aligned}
Au_k \to v \Rightarrow \|Au_k - Au_j\| &\to 0, \quad k,j \to \infty \\
\Rightarrow \|u_k - u_j\| &\to 0, \quad k,j \to \infty && (*) \\
\Rightarrow u_k &\to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\
\Rightarrow v &= Au && (A \text{ continuous})
\end{aligned}
$$

5. $R(A) = H$:

If not $\exists w \neq 0$ in H with $w \in R(A)^{\perp}$ ($R(A)$ closed)

4. A has closed range. Indeed

$$
\begin{aligned}
Au_k \to v \Rightarrow \|Au_k - Au_j\| &\to 0, \quad k,j \to \infty \\
\Rightarrow \|u_k - u_j\| &\to 0, \quad k,j \to \infty && (*) \\
\Rightarrow u_k \to u \in H, &\quad k \to \infty && (H \text{ Hilbert}) \\
\Rightarrow v = Au & && (A \text{ continuous})
\end{aligned}
$$

5. $R(A) = H$:

If not $\exists w \neq 0$ in H with $w \in R(A)^{\perp}$ ($R(A)$ closed)

But then $\beta \|w\|^2 \leq B[w,w] = (Aw,w)_H = 0$.

4. A has closed range. Indeed

$$
\begin{aligned}
Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty \\
&\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty && (*) \\
&\Rightarrow u_k \to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\
&\Rightarrow v = Au && (A \text{ continuous})
\end{aligned}
$$

5. $R(A) = H$:
If not $\exists w \neq 0$ in H with $w \in R(A)^{\perp}$ ($R(A)$ closed)
But then $\beta \|w\|^2 \leq B[w,w] = (Aw,w)_H = 0$.

6. Riesz $\Rightarrow \exists w$ s.t. $\langle f, v\rangle = (w,v)_H \ \forall v \in H$.

4. A has closed range. Indeed

$$\begin{aligned}
Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty \\
&\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty && (*) \\
&\Rightarrow u_k \to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\
&\Rightarrow v = Au && (A \text{ continuous})
\end{aligned}$$

5. $R(A) = H$:

If not $\exists w \neq 0$ in H with $w \in R(A)^{\perp}$ ($R(A)$ closed)

But then $\beta\|w\|^2 \leq B[w,w] = (Aw,w)_H = 0$.

6. Riesz $\Rightarrow \exists w$ s.t. $\langle f, v\rangle = (w,v)_H \ \forall v \in H$.

Let $u \in H$ with $Au = w$.

4. A has closed range. Indeed

$$\begin{aligned}
Au_k \to v \Rightarrow \|Au_k - Au_j\| &\to 0, \quad k,j \to \infty \\
\Rightarrow \|u_k - u_j\| &\to 0, \quad k,j \to \infty && (*) \\
\Rightarrow u_k &\to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\
\Rightarrow v &= Au && (A \text{ continuous})
\end{aligned}$$

5. $R(A) = H$:
If not $\exists w \neq 0$ in H with $w \in R(A)^\perp$ ($R(A)$ closed)
But then $\beta \|w\|^2 \leq B[w,w] = (Aw,w)_H = 0$.

6. Riesz $\Rightarrow \exists w$ s.t. $\langle f,v \rangle = (w,v)_H \ \forall v \in H$.
Let $u \in H$ with $Au = w$. Then

$$B[u,v] = (Au,v)_H = (w,v)_H = \langle f,v \rangle.$$

4. A has closed range. Indeed

$$\begin{aligned}
Au_k \to v \Rightarrow \|Au_k - Au_j\| &\to 0, \quad k,j \to \infty \\
\Rightarrow \|u_k - u_j\| &\to 0, \quad k,j \to \infty && (*) \\
\Rightarrow u_k \to u \in H, &\quad k \to \infty && (H \text{ Hilbert}) \\
\Rightarrow v = Au & && (A \text{ continuous})
\end{aligned}$$

5. $R(A) = H$:
If not $\exists w \neq 0$ in H with $w \in R(A)^\perp$ ($R(A)$ closed)
But then $\beta\|w\|^2 \leq B[w,w] = (Aw,w)_H = 0$.

6. Riesz $\Rightarrow \exists w$ s.t. $\langle f, v\rangle = (w,v)_H \ \forall v \in H$.
Let $u \in H$ with $Au = w$. Then

$$B[u,v] = (Au,v)_H = (w,v)_H = \langle f, v\rangle.$$

7. Uniqueness:

$$f = 0 \Rightarrow \beta\|u\|^2 \leq B[u,u] = \langle 0, u\rangle = 0$$

4. A has closed range. Indeed

$$\begin{aligned} Au_k \to v &\Rightarrow \|Au_k - Au_j\| \to 0, \quad k,j \to \infty \\ &\Rightarrow \|u_k - u_j\| \to 0, \quad k,j \to \infty && (*) \\ &\Rightarrow u_k \to u \in H, \quad k \to \infty && (H \text{ Hilbert}) \\ &\Rightarrow v = Au && (A \text{ continuous}) \end{aligned}$$

5. $R(A) = H$:
If not $\exists w \neq 0$ in H with $w \in R(A)^\perp$ ($R(A)$ closed)
But then $\beta \|w\|^2 \leq B[w,w] = (Aw,w)_H = 0$.

6. Riesz $\Rightarrow \exists w$ s.t. $\langle f,v\rangle = (w,v)_H \ \forall v \in H$.
Let $u \in H$ with $Au = w$. Then

$$B[u,v] = (Au,v)_H = (w,v)_H = \langle f,v\rangle.$$

7. Uniqueness:

$$\begin{aligned} f = 0 \Rightarrow \beta\|u\|^2 &\leq B[u,u] = \langle 0,u\rangle = 0 \\ &\Rightarrow u = 0. \end{aligned}$$

Energy estimates

$$B[u,v] = \int_U \left(\sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx, \quad u,v \in H_0^1(U).$$

Energy estimates

$$B[u,v] = \int_U \left(\sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx, \quad u,v \in H_0^1(U).$$

Assumptions

- a^{ij}, b^i, $c \in L^\infty(U)$ (bounded coefficients)
- $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$, $a^{ij} = a^{ji}$ (ellipticity)

Energy estimates

$$B[u,v] = \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx, \quad u,v \in H_0^1(U).$$

Assumptions

- a^{ij}, b^i, $c \in L^\infty(U)$ (bounded coefficients)
- $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$, $a^{ij} = a^{ji}$ (ellipticity)

Theorem (Energy estimates)

$\exists \alpha, \beta > 0$ *and* $\gamma \geq 0$ *s.t.*

1. $|B[u,v]| \leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}$
2. $\beta \|u\|_{H_0^1}^2 \leq B[u,u] + \gamma \|u\|_{L^2}^2$

$\forall u, v \in H_0^1(U)$.

Proof.

$$(1) \qquad |B[u,v]| \leq \Big(\sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} + \sum_{i=1}^{n} \|b^i\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} + \|c\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} \Big)$$

Proof.

$$(1) \qquad |B[u,v]| \leq \Big(\sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} + \sum_{i=1}^{n} \|b^i\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} + \|c\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} \Big) \leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1},$$

Proof.

(1)
$$|B[u,v]| \le \Big(\sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} + \sum_{i=1}^{n} \|b^i\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} + \|c\|_{L^\infty} \|u\|_{H_0^1} \|v\|_{H_0^1} \Big)$$
$$\le \alpha \|u\|_{H_0^1} \|v\|_{H_0^1},$$
$$\alpha := \sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty} + \sum_{i=1}^{n} \|b^i\|_{L^\infty} + \|c\|_{L^\infty}.$$

$$\theta \int_U |Du|^2\, dx \leq \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\, dx \tag{2}$$

(2)
$$\theta \int_U |Du|^2\,dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\,dx$$
$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + c u^2 \right) dx$$

$$
\begin{aligned}
(2) \quad \theta \int_U |Du|^2 \, dx &\leq \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} \, dx \\
&= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + c u^2 \right) dx \\
&\leq B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2 .
\end{aligned}
$$

$$(2) \quad \theta \int_U |Du|^2\,dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\,dx$$

$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right) dx$$

$$\le B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2.$$

$$\|Du\|_{L^2} \|u\|_{L^2} \le \frac{\varepsilon}{2} \|Du\|_{L^2}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2.$$

(2) $$\theta \int_U |Du|^2\,dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\,dx$$
$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right) dx$$
$$\le B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2.$$

$$\|Du\|_{L^2} \|u\|_{L^2} \le \frac{\varepsilon}{2} \|Du\|_{L^2}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2.$$

Choose $\varepsilon > 0$ s.t. $\varepsilon \sum \|b^i\|_{L^\infty} < \theta$.

(2) $$\theta \int_U |Du|^2\,dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\,dx$$

$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right) dx$$

$$\le B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2.$$

$$\|Du\|_{L^2} \|u\|_{L^2} \le \frac{\varepsilon}{2} \|Du\|_{L^2}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2.$$

Choose $\varepsilon > 0$ s.t. $\varepsilon \sum \|b^i\|_{L^\infty} < \theta$.

$$\theta \|Du\|_{L^2}^2 \le B[u,u] + \frac{\theta}{2} \|Du\|_{L^2}^2 + \gamma \|u\|_{L^2}^2$$

(2) $$\theta \int_U |Du|^2\,dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\,dx$$
$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right) dx$$
$$\le B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2.$$

$$\|Du\|_{L^2}\|u\|_{L^2} \le \frac{\varepsilon}{2}\|Du\|_{L^2}^2 + \frac{1}{2\varepsilon}\|u\|_{L^2}^2.$$

Choose $\varepsilon > 0$ s.t. $\varepsilon \sum \|b^i\|_{L^\infty} < \theta$.

$$\frac{\theta}{2}\|Du\|_{L^2}^2 \le B[u,u] + \gamma \|u\|_{L^2}^2$$

(2) $$\theta \int_U |Du|^2 \, dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} \, dx$$
$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right) dx$$
$$\le B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2 .$$

$$\|Du\|_{L^2} \|u\|_{L^2} \le \frac{\varepsilon}{2} \|Du\|_{L^2}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2 .$$

Choose $\varepsilon > 0$ s.t. $\varepsilon \sum \|b^i\|_{L^\infty} < \theta$.

$$\frac{\theta}{2} \|Du\|_{L^2}^2 \le B[u,u] + \gamma \|u\|_{L^2}^2$$

Poincaré's ineq $\Rightarrow$ $\|u\|_{H_0^1(U)} \le C \|Du\|_{L^2(U)}$, $u \in H_0^1(U)$.

(2) $$\theta \int_U |Du|^2\,dx \le \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j}\,dx$$
$$= B[u,u] - \int_U \left(\sum_{i=1}^n b^i u_{x_i} u + cu^2 \right) dx$$
$$\le B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \|Du\|_{L^2} \|u\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2}^2.$$

$$\|Du\|_{L^2} \|u\|_{L^2} \le \frac{\varepsilon}{2} \|Du\|_{L^2}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2.$$

Choose $\varepsilon > 0$ s.t. $\varepsilon \sum \|b^i\|_{L^\infty} < \theta$.

$$\frac{\theta}{2} \|Du\|_{L^2}^2 \le B[u,u] + \gamma \|u\|_{L^2}^2$$

Poincaré's ineq $\Rightarrow \|u\|_{H_0^1(U)} \le C\|Du\|_{L^2(U)}$, $u \in H_0^1(U)$.

$$\Rightarrow \beta \|u\|_{H_0^1}^2 \le B[u,u] + \gamma \|u\|_{L^2}^2.$$

$\square$

Existence

Theorem (Existence)

$\exists \gamma \geq 0$ s.t. $\forall \mu \geq \gamma$ and $f \in L^2(U)$ $\exists$ unique weak sol. $u \in H_0^1(U)$ of

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = 0 \text{ on } \partial U. \end{cases}$$

Existence

Theorem (Existence)

$\exists \gamma \geq 0$ s.t. $\forall \mu \geq \gamma$ and $f \in L^2(U)$ $\exists$ unique weak sol. $u \in H_0^1(U)$ of

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = 0 \text{ on } \partial U. \end{cases}$$

Proof.

Let γ be as in the previous thm. and

$$B_\mu[u,v] := B[u,v] + \mu(u,v)_{L^2}$$

(bilinear form for $L + \mu$).

$\square$

Existence

Theorem (Existence)

$\exists \gamma \geq 0$ s.t. $\forall \mu \geq \gamma$ and $f \in L^2(U)$ $\exists$ unique weak sol. $u \in H_0^1(U)$ of

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = 0 \text{ on } \partial U. \end{cases}$$

Proof.

Let γ be as in the previous thm. and

$$B_\mu[u,v] := B[u,v] + \mu(u,v)_{L^2}$$

(bilinear form for $L + \mu$).

Lax-Milgram $\Rightarrow \exists! u \in H_0^1$ s.t. $B_\mu[u,v] = (f,v)_{L^2}$. $\square$

Remark

If $b^i = 0 \; \forall i$ and $c \geq 0$, then the energy estimate holds with $\gamma = 0$, so we get existence for the original problem

$$\begin{cases} Lu = f \text{ in } U, \\ \;\; u = 0 \text{ on } \partial U. \end{cases}$$

Can also be seen directly using Riesz' repr. thm. (exercise)
More general results: Evans, 6.2.3

Remark

If $b^i = 0 \ \forall i$ and $c \geq 0$, then the energy estimate holds with $\gamma = 0$, so we get existence for the original problem

$$\begin{cases} Lu = f \text{ in } U, \\ \ u = 0 \text{ on } \partial U. \end{cases}$$

Can also be seen directly using Riesz' repr. thm. (exercise)
More general results: Evans, 6.2.3

Remark

We could also consider

$$B[u,v] = \langle f, v \rangle$$

with $f \in (H_0^1(U))^*$.

Evans, 5.9.1:
$(H_0^1(U))^* = H^{-1}(U) = \{f := f^0 - \sum_{i=1}^n f^i_{x_i}, \quad f^i \in L^2(U)\}$.

Non-homogeneous BCs

$$\begin{cases} Lu = f \text{ in } U, \\ \quad u = g \text{ on } \partial U \end{cases}$$

Non-homogeneous BCs

$$\begin{cases} Lu = f \text{ in } U, \\ \quad u = g \text{ on } \partial U \end{cases}$$

Suppose $\exists w \colon U \to \mathbb{R}$ s.t. $w = g$ on ∂U.

Then $\tilde{u} := u - w$ satisfies

$$\begin{cases} L\tilde{u} = \tilde{f} \text{ in } U, \\ \quad \tilde{u} = 0 \text{ on } \partial U \end{cases}$$

with $\tilde{f} = f - Lw$.

Non-homogeneous BCs

$$\begin{cases} Lu = f \text{ in } U, \\ \ \ u = g \text{ on } \partial U \end{cases}$$

Suppose $\exists w \colon U \to \mathbb{R}$ s.t. $w = g$ on ∂U.

Then $\tilde{u} := u - w$ satisfies

$$\begin{cases} L\tilde{u} = \tilde{f} \text{ in } U, \\ \ \ \tilde{u} = 0 \text{ on } \partial U \end{cases}$$

with $\tilde{f} = f - Lw$.

Theorem

Let U be bdd with ∂U C^1. $\exists$ bdd, surjective, linear operator $T \colon H^1(U) \to L^2(\partial U)$, s.t. $Tu = u|_{\partial\Omega}$ if $u \in C(\overline{U})$.

Non-homogeneous BCs

$$\begin{cases} Lu = f \text{ in } U, \\ \quad u = g \text{ on } \partial U \end{cases}$$

Suppose $\exists w \colon U \to \mathbb{R}$ s.t. $w = g$ on ∂U.

Then $\tilde{u} := u - w$ satisfies

$$\begin{cases} L\tilde{u} = \tilde{f} \text{ in } U, \\ \quad \tilde{u} = 0 \text{ on } \partial U \end{cases}$$

with $\tilde{f} = f - Lw$.

Theorem

Let U be bdd with ∂U C^1. $\exists$ bdd, surjective, linear operator $T \colon H^1(U) \to L^2(\partial U)$, s.t. $Tu = u|_{\partial\Omega}$ if $u \in C(\overline{U})$.

Theorem

Let U be bdd with ∂U C^1. $H_0^1(U) = \{u \in H^1(U) \colon Tu = 0\}$.

Theorem

Let L be as in the previous existence result, $f \in L^2(U)$, $g \in L^2(\partial U)$. $\exists$ unique weak sol. $u \in H^1(U)$ of the BVP

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = g \text{ on } \partial U \end{cases}$$

for all $\mu \geq \gamma$.

Theorem

Let L be as in the previous existence result, $f \in L^2(U)$, $g \in L^2(\partial U)$. $\exists$ unique weak sol. $u \in H^1(U)$ of the BVP

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = g \text{ on } \partial U \end{cases}$$

for all $\mu \geq \gamma$.

Here weak sol. means $B[u,v] = (f,v)_{L^2}\ \forall v \in H_0^1(U)$ and $Tu = g$.

Theorem

Let L be as in the previous existence result, $f \in L^2(U)$, $g \in L^2(\partial U)$. $\exists$ unique weak sol. $u \in H^1(U)$ of the BVP

$$\begin{cases} Lu + \mu u = f \text{ in } U, \\ \qquad\quad u = g \text{ on } \partial U \end{cases}$$

for all $\mu \geq \gamma$.

Here weak sol. means $B[u,v] = (f,v)_{L^2}\ \forall v \in H_0^1(U)$ and $Tu = g$.

Remark: Requires solvability for $\tilde{f} \in H^{-1}(U)$, since $Lw \in H^{-1}(U)$ if $w \in H_0^1(U)$.