LUND UNIVERSITY

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SOLUTIONS, APRIL 2

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5. Note that $e^{-|x|} \in L^1(\mathbb{R})$ so that the Fourier transform is well-defined in the classical sense (as an integral). We have

$$\begin{aligned}\mathscr{F}(e^{-|x|}) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}e^{-|x|}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(1-i\xi)x}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(1+i\xi)x}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-i\xi}+\frac{1}{1+i\xi}\right) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+\xi^2}.\end{aligned}$$

Fourier's inversion formula tells us that

$$e^{-|x|} = \mathscr{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\frac{2}{1+\xi^2}\right)$$

where the right hand side can be interpreted either in the distributional sense or using the L^2-definition of the inverse Fourier transform. However, since $\frac{1}{\sqrt{2\pi}}\frac{2}{1+\xi^2}$ belongs to $L^1(\mathbb{R})$, the right hand side also makes sense as a classical Fourier integral. This means that we have the equality

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{2}{1+\xi^2}e^{ix\xi}\,\mathrm{d}\xi$$

for each x. Evaluating at $x=0$ gives

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{1}{1+\xi^2}\,\mathrm{d}\xi = \pi$$

(this can of course also be deduced by using the primitive function $\arctan x$). Using Plancherel's theorem, we get that

$$\int_{-\infty}^{\infty} e^{-2|x|}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{(1+\xi^2)^2}\,\mathrm{d}\xi$$

and simplifying this gives

$$\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^2}\,\mathrm{d}x = \pi\int_{0}^{\infty}e^{-2x}\,\mathrm{d}x = \frac{\pi}{2}.$$

6. Assuming that u is sufficiently regular, we get that

$$\hat{u}''(\xi,y) = \xi^2\hat{u}(\xi,y),$$

where the primes denote derivation with respect to y and the hats Fourier transformation in x. From this we get that

$$\hat{u}(\xi,y) = a(\xi)e^{-|\xi|y} + b(\xi)e^{|\xi|y}$$

Please, turn over!

for some $a(\xi)$ and $b(\xi)$. In order to obtain a bounded solution for $y > 0$ we set $b(\xi) = 0$, in which case $a(\xi) = \hat{g}(\xi)$. Thus,

$$\hat{u}(\xi, y) = \hat{g}(\xi) e^{-|\xi| y},$$

which in turn gives

$$u(x,y) = \mathscr{F}^{-1}(\hat{g}(\xi) e^{-|\xi| y}) = \frac{1}{\sqrt{2\pi}} g * \mathscr{F}^{-1}(e^{-|\xi| y}).$$

By the previous exercise, we have

$$\mathscr{F}^{-1}(e^{-|\xi| y})(x) = \mathscr{F}(e^{-|\xi| y})(-x) = \frac{1}{y} \mathscr{F}(e^{-|\xi|}) \left(-\frac{x}{y} \right) = \frac{1}{\sqrt{2\pi}} \frac{2}{y(1+\frac{x^2}{y^2})} = \frac{1}{\sqrt{2\pi}} \frac{2y}{x^2+y^2}$$

and hence

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-s)^2+y^2} g(s)\, \mathrm{d}s.$$

This coincides with Poisson's formula for the upper half plane in Evans.

7. Denote the terms in the second series in (1.9) by $u_j(x,t)$. Since $\|u_j\|_{L^\infty(\mathbb{R}^2)} \leq \frac{4}{\pi(2j+1)^2}$, the series converges uniformly on $\mathbb{R}^2$ to a continuous function by Weierstrass' M-test. Moreover, each function $u_j(x,t)$ is a classical solution of the wave equation. It follows by linearity that each partial sum

$$s_n(x,t) = \sum_{j=0}^{n} u_j(x,t)$$

solves the wave equation and that $s_n(x,t)$ converges uniformly on $\mathbb{R}^2$ (and hence also in $\mathscr{D}'(\mathbb{R}^2)$) to $u(x,t)$. But then for each $\varphi \in \mathscr{D}(\mathbb{R}^2)$ we have that

$$\begin{aligned}
\langle (\partial_t^2 - c^2 \partial_x^2) u, \varphi \rangle &= \langle u, (\partial_t^2 - c^2 \partial_x^2) \varphi \rangle \\
&= \langle \lim_{n\to\infty} s_n, (\partial_t^2 - c^2 \partial_x^2) \varphi \rangle \\
&= \lim_{n\to\infty} \langle s_n, (\partial_t^2 - c^2 \partial_x^2) \varphi \rangle \\
&= \lim_{n\to\infty} \langle (\partial_t^2 - c^2 \partial_x^2) s_n, \varphi \rangle \\
&= 0.
\end{aligned}$$

In other words, u solves the wave equation $\partial_t^2 u = c^2 \partial_x^2 u$ in the sense of distributions.

10. **a)** The associated quadratic form is $q(\xi) = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3$. Completing the squares, we get

$$q(\xi) = (\xi_1 + \tfrac{1}{2}\xi_2 + \tfrac{1}{2}\xi_3)^2 + \tfrac{3}{4}\xi_2^2 + \tfrac{3}{4}\xi_3^2 - \tfrac{1}{2}\xi_1\xi_3 = (\xi_1 + \tfrac{1}{2}\xi_2 + \tfrac{1}{2}\xi_3)^2 + \tfrac{3}{4}(\xi_2 - \tfrac{1}{3}\xi_3)^2 + \tfrac{2}{3}\xi_3^2.$$

Hence, q is positive definite and the equation is elliptic.
Alternatively, we can find the eigenvalues of the associated symmetric matrix

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

These are 1 and $1 \pm 1/\sqrt{2}$. Again, we see that the quadratic form is positive definite since the eigenvalues are positive.

b) The quadratic form $q(\xi) = \xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2$ is already diagonalised. We see that there are two positive squares and two negative squares. Hence, the equation is neither elliptic, hyperbolic, nor parabolic. In fact, this is an example of an ultrahyperbolic equation.

c) The quadratic form $q(\xi) = \xi_1^2 + \xi_2^2 + 4\xi_1\xi_2$ can be rewritten as

$$q(\xi) = (\xi_1 + 2\xi_2)^2 - 3\xi_2^2.$$

Since there is one positive square and one negative, we see that the equation is hyperbolic.

11. We have that

$$\begin{aligned}\langle(\partial_t^2 - \partial_x^2)\Phi, \varphi\rangle &= \langle\Phi, (\partial_t^2 - \partial_x^2)\varphi\rangle \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{|x|}^{\infty}(\varphi_{tt}(x,t) - \varphi_{xx}(x,t))\,\mathrm{d}t\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{0}(\varphi_x(s,-s) - \varphi_t(s,-s))\,\mathrm{d}s - \frac{1}{2}\int_0^{\infty}(\varphi_x(s,s) + \varphi_t(s,s))\,\mathrm{d}s\end{aligned}$$

(integrate the first term with respect to t and the second with respect to x and rearrange the terms). Note that φ has compact support, so that we don't have to worry about the convergence of these integrals. The last expression can be rewritten as

$$\frac{1}{2}\int_{-\infty}^{0}\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s,-s)\,\mathrm{d}s - \frac{1}{2}\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s,s) = \varphi(0,0) = \langle\delta_{(0,0)}, \varphi\rangle,$$

which concludes the proof.

Evans 5.10

4. Let $u \in W^{1,p}(0,1)$, $1 \leq p < \infty$.

a) Denote the weak derivative of u by $\partial_x u$. Then $\partial_x u \in L^p(0,1)$ and hence also in $L^1(0,1)$ (since $(0,1)$ has finite measure). Therefore, there exists a primitive $\tilde{u}$ to $\partial_x u$,

$$\tilde{u}(x) = \int_0^x \partial_x u(t)\,dt.$$

$\tilde{u}$ is absolutely continuous by definition, with almost everywhere defined derivative $\partial_x u$. We claim that $\tilde{u}$ also has $\partial_x u$ as weak derivative. Indeed, this follows from the integration by parts formula for absolutely continuous functions (see e.g. Cohn, Corollary 6.3.9). Thus, $u - \tilde{u}$ has weak derivative 0. We claim that this implies that $u - \tilde{u}$ is constant. Indeed, assume that

$$\int_0^1 f\psi'\,dx = 0$$

for all $\psi \in C_c^\infty(0,1)$. Let φ_0 be a fixed function in $C_c^\infty(0,1)$ with integral 1. Set $\mathscr{I}(\varphi) = \int_0^1 \varphi\,dx$ for a general $\varphi \in C_c^\infty(0,1)$. If $\varphi \in C_c^\infty(0,1)$, then

$$\psi(x) = \int_0^x (\varphi(t) - \mathscr{I}(\varphi)\varphi_0(t))\,dt$$

belongs to $C_c^\infty(0,1)$, since ψ is smooth and is constant for x near 0 and 1 with values

$$\psi(0) = 0$$

and

$$\psi(1) = \mathscr{I}(\varphi) - \mathscr{I}(\varphi)\mathscr{I}(\varphi_0) = 0,$$

respectively. But then

$$0 = \int_0^1 f\psi'\,dx = \int_0^1 f\varphi\,dx - \left(\int_0^1 \varphi\,dx\right)\left(\int_0^1 f\varphi_0\,dx\right) = \int_0^1 (f - \mathscr{I}(f\varphi_0))\varphi\,dx.$$

Hence, $f = \mathscr{I}(f\varphi_0)$ a.e.

b) Assume for simplicity that $y < x$. We have

$$u(x) - u(y) = \int_y^x u'(t)\,dt = \int_y^x 1 \cdot u'(t)\,dt$$

and therefore, by Hölder's inquality,

$$|u(x) - u(y)| \le \left(\int_y^x 1^q\,dt\right)^{\frac{1}{q}} \left(\int_y^x |u'|^p\,dt\right)^{\frac{1}{p}} \le |x-y|^{\frac{1}{q}} \left(\int_0^1 |u'|^p\,dt\right)^{\frac{1}{p}}.$$

Here $1/q = 1 - 1/p$.

9. Assume first that $u \in C_c^\infty(U)$. Then

$$\|Du\|_{L^2}^2 = \int_U Du \cdot Du\,dx = -\int_U u\,\Delta u\,dx \le \|u\|_{L^2}\|\Delta u\|_{L^2} \le C\|u\|_{L^2}\|D^2u\|_{L^2}.$$

If $u \in H^2(U) \cap H_0^1(U)$, we take sequences $\{v_k\} \subset C_c^\infty(U)$ and $\{w_k\} \subset C^\infty(\bar{U})$ with $\|v_k - u\|_{H^1} \to 0$ and $\|w_k - u\|_{H^2} \to 0$ and note that

$$\int_U Dv_k \cdot Dw_k\,dx = -\int_U v_k\,\Delta w_k\,dx$$

whence

$$\int_U Du \cdot Du\,dx = -\int_U u\,\Delta u\,dx.$$

The inequality follows as before.