PDE Lecture

Sobolev spaces

March 31

Definition and basic properties

Weak derivatives

$U \subset \mathbb{R}^n$ open, non-empty. Recall from last time:

Definition

If $u \in L^1_{\text{loc}}(U)$, we say that $v \in L^1_{\text{loc}}(U)$ is the αth weak partial derivative of u, $v = D^\alpha u$, if

$$\int_U u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U v \varphi \, dx \qquad \forall \varphi \in C_c^\infty(U).$$

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Definition

Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(U)$ is defined by

$$W^{k,p}(U) = \{u \in L^p(U) \colon D^\alpha u \in L^p(U) \; \forall \alpha \text{ s.t. } |\alpha| \leq k\}.$$

Norm:

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(U)}\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(U)}, & p = \infty. \end{cases}$$

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$H^k(U) = W^{k,2}(U)$ has inner product

$$(u,v)_{H^k(U)} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L^2(U)} = \sum_{|\alpha|\le k} \int_U D^\alpha u D^\alpha v \, dx$$

and

$$(u,u)_{H^k} = \|u\|_{W^{k,2}}^2$$

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- Then each $\{D^\alpha u_m\}$ is Cauchy in L^p ($\|D^\alpha u\|_{L^p} \leq \|u\|_{W^{k,p}}$).

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- L^p complete $\Rightarrow D^\alpha u_m \to u_\alpha$ in L^p.
- Let $u = u^{(0)}$. Then

$$\int_U u_m D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U D^\alpha u_m \, \varphi \, dx \qquad \forall \varphi \in C_c^\infty(U).$$

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- Hence $u_\alpha = D^\alpha u$ and $D^\alpha u_m \to D^\alpha u$ in L^p for all $|\alpha| \le k$.

□

Additional spaces

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Q: What is $W_0^{0,p}(U)$, $1 \leq p < \infty$?

Example

$U = B^0(0,1)$

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Weak derivative

$$u_{x_i}(x) = -\alpha \frac{x_i}{|x|^{\alpha+2}}.$$

in U if $\alpha + 1 < n$:

$$|Du| = \frac{|\alpha|}{|x|^{\alpha+1}} \in L^1(U) \Leftrightarrow \alpha + 1 < n$$

(see Evans, Example 3, p. 260 for details)

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Generally:

$$|Du| \in L^p(U) \Leftrightarrow (\alpha+1)p < n$$

Hence, $u \in W^{1,p}$ if and only if $\alpha < \frac{n-p}{p}$.

Note: u is unbounded if $\alpha > 0$! This requires $p < n$.

Approximation

Let

$$U_\varepsilon = \{x \in U \colon\ \mathrm{dist}(x, \partial U) > \varepsilon\}$$

and

$$\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x), \quad \varepsilon > 0$$

with $\eta \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\eta \subseteq B(0,1)$, $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$ (Appendix C.5).

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For $x \in U_\varepsilon$,

$$u^\varepsilon(x) = \eta_\varepsilon * u = \int_U \eta_\varepsilon(x-y)u(y)\,dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y)u(x-y)\,dy.$$

The first integral is well-defined for $x \in \mathbb{R}^n$.

Theorem

Assume $u \in W^{k,p}(U)$, $1 \leq p < \infty$ *and set*

$$u^{\varepsilon} = \eta_{\varepsilon} * u \quad \text{in } U_{\varepsilon}.$$

Then

1. $u^{\varepsilon} \in C^{\infty}(U_{\varepsilon})$ *for each* $\varepsilon > 0$,
2. $u^{\varepsilon} \to u$ *in* $W^{k,p}_{loc}(U)$ *as* $\varepsilon \to 0$.

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Proof.
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In fact, extends to $C^\infty(\mathbb{R}^n)$ function, vanishing if $\operatorname{dist}(x,U) \ge \varepsilon$.

2: For $x \in U_\varepsilon$:

$$D^\alpha u^\varepsilon(x) = \int_U D_x^\alpha \eta_\varepsilon(x-y)u(y)\,dy = (-1)^{|\alpha|}\int_U D_y^\alpha \eta_\varepsilon(x-y)u(y)\,dy$$

2: For $x \in U_\varepsilon$:

$$\begin{aligned} D^\alpha u^\varepsilon(x) &= \int_U D_x^\alpha \eta_\varepsilon(x-y)u(y)\,dy = (-1)^{|\alpha|}\int_U D_y^\alpha \eta_\varepsilon(x-y)u(y)\,dy \\ &= \int_U \eta_\varepsilon(x-y)D^\alpha u(y)\,dy \\ &= (\eta_\varepsilon * D^\alpha u)(x) \end{aligned}$$

since $\eta_\varepsilon(x-\cdot) \in C_c^\infty(U)$.

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If $V \subset\subset U$, then

$$\eta_\varepsilon * D^\alpha u \to D^\alpha u$$

in $L^p(V)$ for $|\alpha| \le k$ (see Appendix C.5). $\square$

Global approximation

Theorem

Assume U is bounded and ∂U is C^1. Suppose $u \in W^{k,p}(U)$, $1 \leq p < \infty$. Then $\exists u_m \in C^\infty(\overline{U})$ such that

$$u_m \to u \quad \text{in } W^{k,p}(U).$$

Proof, see Evans, Theorem 3, p. 266.

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Proof.

First approximate by functions with compact support:

$$u_R(x) = \varphi(R^{-1}x)u(x),$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi(x) = 1$, $|x| \leq 1$ and $\varphi(x) = 0$, $|x| \geq 2$.

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$u_R \in W^{k,p}(\mathbb{R}^n)$, $\operatorname{supp} u_R \subseteq B(0, 2R)$ and

$$u_R \to u \quad \text{in } W^{k,p}(\mathbb{R}^n) \quad \text{as } R \to \infty.$$

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Then approximate u_R by

$$u_{R,\varepsilon} = \eta_\varepsilon * u_R. \qquad \square$$

Extensions

Theorem

Assume U bounded, ∂U C^1. Let V be bounded and open, s.t. $U \subset\subset V$. Then $\exists$ bounded linear operator

$$E\colon W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$:

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1. Reduce to the case when $U = \mathbb{R}^n_+$ and u has compact support in $B(0,r) \cap U$ using partition of unity and change of variables.

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$$\bar{u}(x) = \begin{cases} u(x', x_n), & x_n > 0, \\ u(x', -x_n), & x_n < 0. \end{cases}$$

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Remark: For $k \geq 2$ one can use 'higher order' reflection (Evans).

Sobolev inequalities

Gagliardo-Nirenberg-Sobolev inequality

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Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

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Remark: p^* is the only value of q such that an inequality of the form $\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$ can hold. This can be seen by trying $u_\lambda(x) = u(\lambda x)$ (Evans).

Proof

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$p^* = n/(n-1)$.

Consider the case $n = 2$ for simplicity. Then $p^* = 2$

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$$u(x) = \int_{-\infty}^{x_1} u_{x_1}(y_1, x_2)\, dy_1 = \int_{-\infty}^{x_2} u_{x_2}(x_1, y_2)\, dy_2$$

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$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(y_1, x_2)|\, dy_1, \qquad |u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, y_2)|\, dy_2.$$

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$$|u(x)|^2 \le \left(\int_{-\infty}^{\infty} |Du(y_1, x_2)|\, dy_1\right) \left(\int_{-\infty}^{\infty} |Du(x_1, y_2)|\, dy_2\right).$$

$$|u(x)|^2 \leq \left(\int_{-\infty}^{\infty} |Du(y_1, x_2)| \, dy_1 \right) \left(\int_{-\infty}^{\infty} |Du(x_1, y_2)| \, dy_2 \right).$$

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Integrate in x

$$\|u\|_{L^2}^2 = \int_{\mathbb{R}^2} |u(x)|^2 \, dx \leq \left(\int_{\mathbb{R}^2} |Du| \, dx \right)^2 = \|Du\|_{L^1}^2.$$

$$|u(x)|^2 \le \left(\int_{-\infty}^{\infty} |Du(y_1,x_2)|\,dy_1\right)\left(\int_{-\infty}^{\infty} |Du(x_1,y_2)|\,dy_2\right).$$

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$$\|u\|_{L^2}^2 = \int_{\mathbb{R}^2} |u(x)|^2\,dx \le \left(\int_{\mathbb{R}^2} |Du|\,dx\right)^2 = \|Du\|_{L^1}^2.$$

$\underline{1 < p < n}$

Apply previous estimate to $v := |u|^\gamma$, for some $\gamma > 1$.

Then

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} |D|u|^\gamma|\,dx\\ &= \gamma\int_{\mathbb{R}^n} |u|^{\gamma-1}|Du|\,dx\\ &\le \gamma\left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^n} |Du|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

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Set

$$\gamma := \frac{p(n-1)}{n-p} > 1$$

so that

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = \frac{np}{n-p} = p^*.$$

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Then

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^n} |Du|^p\, dx\right)^{\frac{1}{p}}.$$

□

Theorem

Let $U \subset \mathbb{R}^n$, bdd, open, ∂U C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$ with

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

where C is independent of u.

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- Let $\overline{u} \in W^{1,p}_c(\mathbb{R}^n)$ be an extension of u.
- Let $u_m \in C^\infty_c(\mathbb{R}^n)$ with $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$.
- Previous thm $\Rightarrow$ u_m Cauchy in L^{p^*} with

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C'\|D\overline{u}_m\|_{L^p(\mathbb{R}^n)} \leq C'\|\overline{u}_m\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(U)}.$$

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- Passing to the limit, $\overline{u} \in L^{p^*}$ with

$$\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)},$$

thus $\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)}$.

Poincaré's inequality

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Assume $U \subset \mathbb{R}^n$ *open, bounded. Suppose* $u \in W_0^{1,p}(U)$*,* $1 \leq p < n$*. Then*

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- $|U| < \infty \Rightarrow \|u\|_{L^q(U)} \le C_2\|u\|_{L^{p^*}(U)}$, $1 \le q \le p^*$. $C = C_1C_2$.

$\square$

In particular, for $q = p$, we get

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for $u \in W_0^{1,p}(U)$ (note that $p < p^* = pn/(n-p)$).
Sometimes called Poincaré's inequality.

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It also holds if $p \geq n$, since then $u \in W_0^{1,\tilde{p}}$ for any $\tilde{p} < n$ and

$$\|u\|_{L^{\tilde{p}^*}(U)} \leq C\|Du\|_{L^{\tilde{p}}(U)} \leq C'\|Du\|_{L^p(U)}$$

and $p \leq \tilde{p}^* = \tilde{p}n/(n-\tilde{p})$ if $\tilde{p}$ is suff. close to n, making

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Poincaré's inequality implies that the norm $\|Du\|_{L^p(U)}$ is equivalent to $\|u\|_{W^{1,p}(U)}$ on $W_0^{1,p}(U)$ if U is bounded.