PDE Lecture

Weak derivatives, distributions, classification of PDE

March 24

Motivation

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(Evans, Exercise 2.5.23)

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\begin{cases}
u_{tt} = u_{xx}, & 0 < x < \pi, t > 0, \\
u(0,x) = g(x), & 0 < x < \pi, \\
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Sine series solution

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u(x,t) = \sum_{k=1}^{\infty} a_k \cos(kt) \sin(kx)
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where

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Defines a C^2 solution if $g \in C^3$, $g(0) = g''(0) = g(\pi) = g''(\pi) = 0$.

Still converges unif. to a cont. function if $\sum_k |a_k| < \infty$
(e.g. g piecewise C^1, Wahlén: Example 1.6 and Exercise 7)

In what sense are these solutions?

Weak derivatives

(Evans, 5.2)

Integration by parts:

$$\int_U u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U D^\alpha u \, \varphi \, dx, \quad u, \varphi \in C_c^\infty(U),$$

where $U \subset \mathbb{R}^n$ open.

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Definition

If $u \in L^1_{\mathrm{loc}}(U)$, we say that $v \in L^1_{\mathrm{loc}}(U)$ is the αth weak partial derivative of u, $v = D^\alpha u$, if

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Remark

v is unique if it exists: $\int_U v \, \varphi \, dx = 0$, $\forall \varphi \in C_c^\infty(U) \Rightarrow v = 0$ a.e.

(Lemma on p. 257 of Evans, Theorem 3.3 in Wahlén)

Example

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$$\int_0^{\pi} g(x)\varphi'(x)\,dx = \int_0^{\pi/2} g(x)\varphi'(x)\,dx + \int_{\pi/2}^{\pi} g(x)\varphi'(x)\,dx$$

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Example

Is g' weakly differentiable?

$$\int_0^{\pi} g'(x)\varphi'(x)\,dx = \int_0^{\pi/2} \varphi'(x)\,dx - \int_{\pi/2}^{\pi} \varphi'(x)\,dx = 2\varphi(\pi/2).$$

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for some $v \in L^1_{\mathrm{loc}}(0,\pi)$.

Take $\varphi_m(x)$ with, $0 \le \varphi_m(x) \le 1$, $\varphi_m(\pi/2) = 1$ and $\varphi_m(x) \to 0$, $x \neq \pi/2$ as $m \to \infty$.

See also Evans, Example 2, p. 257.

Distributions

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Note:

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are all linear functionals on $C_c^\infty(U)$. Take $U = \mathbb{R}^n$ for simplicity.

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Definition

A distribution ℓ is a linear functional on $\mathscr{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n)$, which is continuous in the following sense. If $\{\varphi_k\} \subset \mathscr{D}(\mathbb{R}^n)$ satisfies

1. $\exists K \subset \mathbb{R}^n$, compact, s.t. $\operatorname{supp} \varphi_k \subseteq K\ \forall k$, and
2. $\exists \varphi \in \mathscr{D}(\mathbb{R}^n)$ s.t. $D^\alpha \varphi_k \to D^\alpha \varphi$ uniformly (on K) $\forall \alpha$,

then $\ell(\varphi_k) \to \ell(\varphi)$ as $k \to \infty$.
The vector space of distributions is denoted $\mathscr{D}'(\mathbb{R}^n)$.

Example

Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then

$$\ell_u(\varphi) = \langle u, \varphi \rangle := \int_{\mathbb{R}^n} u(x)\, \varphi(x)\, dx, \quad \varphi \in \mathscr{D}(\mathbb{R}^n)$$

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Continuity:

$$|\ell_u(\varphi)| \leq \|u\|_{L^1(K)} \|\varphi\|_{L^\infty(K)}$$

if $\operatorname{supp} \varphi \subseteq K$.

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We write $\langle u, \varphi \rangle$ for $u(\varphi)$ even when $u \in \mathscr{D}'(\mathbb{R}^n)$

Example

Let $a \in \mathbb{R}^n$. Define the Dirac delta distribution at a by

$$\delta_a(\varphi) = \langle \delta_a, \varphi \rangle := \varphi(a).$$

Continuity: $|\delta_a(\varphi)| \leq \|\varphi\|_{L^\infty}$

Operations on distributions

For $h \in \mathbb{R}^n$, let

$$\tau_h \colon \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n), \quad (\tau_h \varphi)(x) = \varphi(x-h).$$

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Define

$$\tau_h \colon \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n), \quad \langle \tau_h u, \varphi \rangle = \langle u, \tau_{-h}\varphi \rangle.$$

Generally, assume that the linear operator

$$L\colon \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n)$$

has a continuous transpose

$$L^T\colon \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n),$$

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Definition

The distributional derivative of $u \in \mathscr{D}'(\mathbb{R}^n)$ is defined by

$$\langle D^\alpha u, \varphi \rangle := \langle u, (-1)^{|\alpha|} D^\alpha \varphi \rangle.$$

Remark: All distributions are infinitely differentiable in this sense.

Example

Recall the function

$$g'(x) = \begin{cases} 1, & 0 \le x < \pi/2, \\ -1, & \pi/2 \le x \le \pi. \end{cases}$$

from before, which was not weakly differentiable:

$$\int_0^\pi g'(x)\varphi'(x)\,dx = \int_0^{\pi/2} \varphi'(x)\,dx - \int_{\pi/2}^{\pi} \varphi'(x)\,dx = 2\varphi(\pi/2).$$

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Answer: $g'' = -2\delta_{\pi/2}$, since

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Question: Can we see this graphically?

What is the distributional derivative of the function u in Example 2 on p. 257 in Evans?

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$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1, \\ 2 & \text{if } 1 < x < 2. \end{cases}$$

Convolution

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where $\mathscr{R}(\varphi)(x) = \varphi(-x)$.

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Thus we define

$$\langle \varphi * u, \psi \rangle = \langle u, \mathscr{R}(\varphi) * \psi \rangle,$$

when $\varphi, \psi \in \mathscr{D}(\mathbb{R}^n)$ and $u \in \mathscr{D}'(\mathbb{R}^n)$.

There is also another way to define convolution,

$$(\varphi * u)(x) = \langle u, \varphi(x - \cdot) \rangle,$$

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Wahlén, Lemma 3.27: These definitions agree. Moroever,

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Example

$(\varphi * \delta_0)(x) = \langle \delta_0, \varphi(x - \cdot) \rangle = \varphi(x)$.

Fourier transform and tempered distributions

$$\int_{\mathbb{R}^n} \hat{u} v \, dx = \int_{\mathbb{R}^n} u \hat{v} \, dx \quad \forall u, v \in \mathscr{S}(\mathbb{R}^n)$$

so $\mathscr{F}^T = \mathscr{F}$.

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The space of tempered distributions is denoted $\mathscr{S}'(\mathbb{R}^n)$.

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Examples

- $\delta_a \in \mathscr{S}'$

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Problem: $\mathscr{F}(\mathscr{D}(\mathbb{R}^n)) \not\subseteq \mathscr{D}(\mathbb{R}^n)$.

Definition

A continuous linear functional on $\mathscr{S}(\mathbb{R}^n)$ is called a tempered distribution.
The space of tempered distributions is denoted $\mathscr{S}'(\mathbb{R}^n)$.

Remark: $\mathscr{D} \subset \mathscr{S} \Rightarrow \mathscr{S}' \subset \mathscr{D}'$.

Examples

- $\delta_a \in \mathscr{S}'$
- $f \in L^1_{\mathrm{loc}}$ and $(1+|x|)^{-N} f \in L^1$ for some $N \Rightarrow f \in \mathscr{S}'$.

Fourier transform and tempered distributions

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$\mathscr{F}, \mathscr{F}^{-1}\colon \mathscr{S}' \to \mathscr{S}'$ are defined by

$$\langle \mathscr{F}(u), \varphi \rangle = \langle u, \mathscr{F}(\varphi) \rangle, \quad \langle \mathscr{F}^{-1}(u), \varphi \rangle = \langle u, \mathscr{F}^{-1}(\varphi) \rangle,$$

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So

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Theorem

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Proof.

$\langle \mathscr{F}\mathscr{F}^{-1}(u), \varphi \rangle = \langle u, \mathscr{F}\mathscr{F}^{-1}(\varphi) \rangle = \langle u, \varphi \rangle$. $\square$

Weak and distributional solutions to PDE

Definition

Given a linear partial differential operator with constant coefficients

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C}$$

and $f \in \mathscr{D}'(\mathbb{R}^n)$, we say that $u \in \mathscr{D}'(\mathbb{R}^n)$ solves the equation

$$P(D)u = f$$

<u>in the sense of distributions</u> if this holds as an equality in $\mathscr{D}'(\mathbb{R}^n)$, that is,

$$\langle u, P(D)^T \varphi \rangle = \langle f, \varphi \rangle, \quad \forall \varphi \in \mathscr{D}(\mathbb{R}^n),$$

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If f and u are both functions, we call u a weak solution.

Example

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

is a distributional solution of

$$u' = \delta_0.$$

Example

Let $G \in L^1_{\text{loc}}(\mathbb{R})$. Then $u(x,t) = G(x-t)$ is a distributional solution of

$$u_t + u_x = 0$$

since if $\varphi \in \mathscr{D}(\mathbb{R}^2)$, then

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Fundamental solutions

Definition

Given a linear partial differential operator

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C}$$

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Proof.

$P(D)(\Phi * f) = (P(D)\Phi) * f = \delta_0 * f = f$. □

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$$\begin{aligned} \langle -\Delta\Phi, \varphi\rangle = \langle \Phi, -\Delta\varphi\rangle &= -\int_{\mathbb{R}^n} \Phi(x)\Delta\varphi(x)\, dx \\ &= \varphi(0) = \langle \delta_0, \varphi\rangle \end{aligned}$$

(Evans, Section 2.2, Theorem 1).

Example

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0, \\ 0, & t < 0. \end{cases}$$

is a fundamental solution for the heat operator $\partial_t - \Delta$.
See pp. 41–42 in Wahlén.

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Not true for variable coefficients, H. Lewy 1957.

L. Hörmander & N. Dencker in Lund have proved necessary and sufficient conditions for solvability of

$$P(x, D)u = f.$$

Classification of PDE

2nd order linear PDE with const. coeff.

$$P(D)u = \sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum a_i \frac{\partial u}{\partial x_i} + a_0 u = f,$$

with $a_{ij} = a_{ij} \in \mathbb{R}$, $a_i \in \mathbb{R}$.

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The equation is called

- elliptic if q is pos. definite
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Examples

- the Laplace equation is elliptic
- the wave equation is hyperbolic
- the heat equation is parabolic

Generalisation

Definition

$$P(D)u = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$$

is elliptic if

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$\widehat{P(D)u} = p(\xi)\hat{u}$

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See Wahlén, Thm. 4.7, for a result in this direction.