LUND UNIVERSITY

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SOLUTIONS, MARCH 13

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1. We make the Ansatz $u(x,y) = \varphi(x)\psi(y)$ and look for a solution which satisfies the boundary conditions $u(0,y) = u(1,y) = 0$. Substituting u into Laplace's equation gives

$$\varphi''(x)\psi(y) + \varphi(x)\psi''(y) = 0$$

and hence

$$\frac{\varphi''(x)}{\varphi(x)} = -\frac{\psi''(y)}{\psi(y)} = -\lambda,$$

for some constant $\lambda \in \mathbb{R}$ (assuming $\varphi(x)\psi(y) \neq 0$). Hence, $-\varphi'' = \lambda\varphi$ and $\psi'' = \lambda\psi$ The boundary conditions on the lateral sides give $\varphi(0) = \varphi(1) = 0$, whence $\lambda = (k\pi)^2$ for some $k = 1,2,3,\ldots$ and $\varphi(x) = A\sin(k\pi x)$ for some $A \in \mathbb{R}$. This in turn gives

$$\psi(y) = B\cosh(k\pi y) + C\sinh(k\pi y)$$

for some constants B and C. Now we look for a solution satisfying the boundary conditions $u_y(x,0) = 0$ and $u(x,1) = x^2 - x$ by making the Ansatz

$$u(x,y) = \sum_{k=1}^{\infty} (B_k\cosh(k\pi y) + C_k\sinh(k\pi y))\sin(k\pi x)$$

(A can be swallowed into B and C). Evaluating at $y = 0$ gives

$$u_y(x,0) = \sum_{k=1}^{\infty} k\pi C_k\sin(k\pi x) = 0,$$

so that $C_k = 0$ for each k. Evaluating at $y = 1$ gives

$$u(x,1) = \sum_{k=1}^{\infty} B_k\cosh(k\pi)\sin(k\pi x) = x^2 - x.$$

In order to find B_k, we expand the function $x^2 - x$ in a sine series. We have that

$$x^2 - x = \sum_{k=1}^{\infty} c_k\sin(k\pi x),$$

where

$$\begin{aligned} c_k &= 2\int_0^1 (x^2 - x)\sin(k\pi x)\,\mathrm{d}x = \frac{2}{k\pi}\int_0^1 (2x-1)\cos(k\pi x)\,\mathrm{d}x \\ &= -\frac{4}{k^2\pi^2}\int_0^1 \sin(k\pi x)\,dx = \begin{cases} -\frac{8}{k^3\pi^3}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \end{aligned}$$

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It follows that

$$u(x,y) = -\sum_{j=0}^{\infty} \frac{8}{(2j+1)^3\pi^3} \frac{\sin((2j+1)\pi x)\cosh((2j+1)\pi y)}{\cosh((2j+1)\pi)}.$$

The calculations above were formal, but using Weierstrass M-test, one sees that the series converges uniformly on the whole square and that one can differentiate it once termwise on the square $[0,1]\times[0,1]$ and infinitely many times on the square $[0,1]\times[0,1)$ (note that $\cosh((2j+1)\pi y)/\cosh((2j+1)\pi) \to 0$ exponentially fast for $y \in [0,1)$). Thus the series defines a function $u \in C^\infty([0,1]\times[0,1)) \cap C^1([0,1]\times[0,1])$ which solves Laplace's equation and the boundary conditions. Note that we cannot hope to do much better since if u were C^2 on the whole closed square, we would get the contradiction $u_{xx}(0,1) = 2 = -u_{yy}(0,1) = 0$, where the boundary conditions and Laplace's equation have been used.

2. We first write Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Making the Ansatz $u(r,\theta) = \varphi(r)\psi(\theta)$ leads to the equations

$$\psi''(\theta) = -\lambda\psi(\theta)$$

and

$$r^2\varphi'' + r\varphi' = \lambda\varphi.$$

Assuming that u is 2π-periodic in θ gives $\lambda = k^2$, $k \in \mathbb{Z}$, and $\psi_k(\theta) = e^{\mathrm{i}k\theta}$ (this is a convenient way of making sure that each solution is only counted once). Inserting $\lambda = k^2$ in the equation for φ gives

$$r^2\varphi'' + r\varphi' = k^2\varphi.$$

This is an Euler equation with linearly independent solutions $r^{|k|}$, $r^{-|k|}$ if $k \neq 0$ and 1, $\log r$ if $k = 0$. Hence, if we require u to be bounded at the origin, we obtain the solutions $u_k(r,\theta) = r^{|k|}e^{\mathrm{i}k\theta}$, $k \in \mathbb{Z}$. We now try to find a solution of Laplace's equation satisying the boundary condition $u(1,\theta) = g(\theta)$ by making the Ansatz

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{\mathrm{i}k\theta}.$$

This gives

$$g(\theta) = u(1,\theta) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}k\theta},$$

so that

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-\mathrm{i}k\theta}\,\mathrm{d}\theta.$$

Hence,

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)e^{-\mathrm{i}ks}\,\mathrm{d}s\right) r^{|k|}e^{\mathrm{i}k\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\sum_{k=-\infty}^{\infty} r^{|k|}e^{\mathrm{i}k(\theta-s)}\,\mathrm{d}s.$$

Moreover,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} r^{|k|}e^{\mathrm{i}k(\theta-s)} &= \sum_{k=0}^{\infty}(re^{i(\theta-s)})^k + \sum_{k=1}^{\infty}(re^{-i(\theta-s)})^k = \frac{1}{1-re^{\mathrm{i}(\theta-s)}} + \frac{re^{-\mathrm{i}(\theta-s)}}{1-re^{-\mathrm{i}(\theta-s)}}\\ &= \frac{1-r^2}{|1-re^{\mathrm{i}(\theta-s)}|^2} = \frac{1-r^2}{|e^{\mathrm{i}s}-re^{\mathrm{i}\theta}|^2}.\end{aligned}$$

Hence,

$$u(r,\theta) = \int_{-\pi}^{\pi} g(s) \frac{1-r^2}{2\pi|e^{\mathrm{i}s} - re^{\mathrm{i}\theta}|^2},$$

and

$$\frac{1-r^2}{2\pi|e^{\mathrm{i}s} - re^{\mathrm{i}\theta}|^2} = K(re^{\mathrm{i}\theta}, e^{\mathrm{i}s}),$$

where K is the Poisson kernel for the unit disc (see Evans Section 2.2.4). The denominator can also be written $2\pi(1 - 2r\cos(\theta - s) + r^2)$.

3. Making the Ansatz $u(x,t) = \varphi(x)\psi(t)$ leads to $\mathrm{i}\varphi(x)\psi'(t) + \varphi''(x)\psi(t) = 0$ and hence

$$\frac{\varphi''(x)}{\varphi(x)} = -\frac{\mathrm{i}\psi'(t)}{\psi(t)} = -\lambda.$$

The periodic boundary conditions give $\varphi(x) = Ae^{\mathrm{i}kx}$ and $\lambda = k^2$, $k \in \mathbb{Z}$ (where we've allowed negative k but haven't written the solutions $e^{-\mathrm{i}kx}$ just like in the previous exercise in order to count each solution just once). We also obtain $\psi(t) = Be^{-ik^2 t}$. We now make the Ansatz

$$u(x,t) = \sum_{k=-\infty}^{\infty} c_k e^{-\mathrm{i}k^2 t} e^{\mathrm{i}kx}$$

in order to find a solution which also satisfies the initial condition. This gives

$$g(x) = u(x,0) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}kx}$$

so that

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-\mathrm{i}kx} \, \mathrm{d}x$$

are the usual complex Fourier coefficients of g. Assuming that g is 2π-periodic and smooth, one can integrate by parts arbitrarily many times in the formula for c_k and obtain that $|c_k| \le C_N(1+|k|)^{-N}$ for any N, where C_N is a constant depending on N. From this it is easily seen that u defined as above indeed is a solution of the initial/boundary-value problem for the Schrödinger equation. Note also that

$$\|u(\cdot,t)\|^2_{L^2(-\pi,\pi)} = 2\pi \sum_{k=-\infty}^{\infty} |c_k e^{-\mathrm{i}k^2 t}|^2 = 2\pi \sum_{k=-\infty}^{\infty} |c_k|^2 = \|g\|^2_{L^2(-\pi,\pi)}$$

by Parseval's formula. Alternatively, one can deduce this using the energy method assuming only that u is a smooth solution of the initial/boundary-value problem. The advantage of the latter is that it also gives uniqueness (without assuming that u is given by the above Fourier series formula).

4. Alternative 1: We use an energy method. Note that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{\pi} u^2(x,t) \, \mathrm{d}x &= \int_0^{\pi} u(x,t) u_t(x,t) \, \mathrm{d}x \\
&= \int_0^{\pi} u(x,t) u_{xx}(x,t) \, \mathrm{d}x \\
&= -\int_0^{\pi} u_x^2(x,t) \, \mathrm{d}x + [u(x,t)u_x(x,t)]_{x=0}^{x=\pi} \\
&= -\int_0^{\pi} u_x^2(x,t) \, \mathrm{d}x \\
&\le 0
\end{aligned}$$

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since $u_x(0,t) = u_x(\pi,t) = 0$. It follows that

$$\int_0^\pi u^2(x,t)\,\mathrm{d}x \le \int_0^\pi u^2(0,t)\,\mathrm{d}x = 0$$

for all $t \in [0,T]$. Hence, $u(x,t) \equiv 0$.

Alternative 2: We use a maximum principle argument. Extend u to an even function on $[-\pi,\pi]\times[0,T]$ by setting $u(x,t) = u(-x,t)$ for $x \in [-\pi,0]$ and then to a 2π-periodic function on $\mathbb{R}\times[0,T]$. The extended function (also denoted u) belongs to $C_1^2(\mathbb{R}\times(0,T])\cap C(\mathbb{R}\times[0,T])$ and solves the heat equation with $u(x,0) = 0$. Since u is bounded (so that $|u(x,t)| \le Ae^{a|x|^2}$ for some $A, a > 0$), one can in fact deduce directly from Theorem 7 in Evans Section 2.3 that u vanishes. However, one can also see this from the usual maximum principle by regarding the domain $[-\pi,\pi]$ with periodic boundary conditions as a compact manifold without boundary (the unit circle). This means that the points $(-\pi,t)$ and (π,t) for $t \in (0,T)$ can be regard as interior points. Similarly, the points $(-\pi,T)$ and (π,T) can be regarded as interior points relative to the set $\mathbb{R}\times\{T\}$. The details are left to the reader.

Evans 4.7

1. Make the Ansatz $u = v(x_1) + w(x_2)$. Inserting this in the equation, we get

$$(v'(x_1))^2 v''(x_1) = -(w'(x_2))^2 w''(x_2).$$

Since the left hand side is independent of x_2 and the right hand side independent of x_1, both sides must be constant. Call this constant $a \in \mathbb{R}$ (we look for real solutions). Using the chain rule, we get

$$\frac{d}{dx_1}(v'(x_1))^3 = 3a$$

and hence

$$v'(x_1) = (3ax_1 + b_1)^{\frac{1}{3}}$$

for some constant b_1. Finally, assuming $a \neq 0$, we get

$$v(x_1) = \frac{1}{4a}(3ax_1 + b_1)^{\frac{4}{3}} + c_1$$

for some constant c_1. If $a = 0$, on the other hand, we get

$$v(x_1) = b_1^{\frac{1}{3}} x_1 + c_1.$$

Noting that $w(x) = -v(x)$ solves $(w')^2 w'' = -a$ if v solves $(v')^2 v'' = a$, we similarly get

$$w(x_2) = -\frac{1}{4a}(3ax_2 + b_2)^{\frac{4}{3}} - c_2$$

if $a \neq 0$ and

$$w(x_2) = -b_2^{\frac{1}{3}} x_2 - c_2$$

if $a = 0$.

In the case $a = 0$ we therefore obtain the solution

$$u(x_1,x_2) = B_1 x_1 + B_2 x_2 + C,$$

where we have set $B_j = b_j^{\frac{1}{3}}$, $j = 1,2$, and $C = c_1 - c_2$. This is of course a smooth solution on $\mathbb{R}^2$.

In the case $a \neq 0$ we choose $b_1 = b_2 = 0$, which just corresponds to a translation. The solution can then be written

$$u(x_1,x_2) = A(x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}) + C$$

with $A = \frac{3^{\frac{4}{3}} a^{\frac{1}{3}}}{4}$. Note that $x_j^{\frac{4}{3}}$ is a well-defined C^1 function on the whole real line. However, it isn't twice differentiable at $x_1 = 0$. This u is therefore only a classical solution of the equation away from the coordinate axes.

2. To find the solution one makes the Ansatz $u(x_1,x_2) = v(x_1)w(x_2)$. One then obtains the equations $v''(x_1) = -\lambda v(x_1)$, $w''(x_2) = \lambda w(x_2)$. Furthermore, the condition $u(x_1,0) = 0$ suggests taking $w(0) = 0$, while the condition $u_{x_2}(x_1,0) = \sin(nx_1)/n$ suggest taking $\lambda = n^2$, $v(x_1) = \sin(nx_1)/n^2$ and $w(x_2) = \sinh(nx_2)$, resulting in

$$u(x_1,x_2) = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2).$$

One easily verifies that this is indeed a solution.

Note that the initial data converge to 0 both uniformly and in L^2_{per} as $n \to \infty$. However, for each fixed $x_2 > 0$ the function $x_1 \to u(x_1,x_2)$ tends to infinity, both in the supremum norm and in L^2, since

$$u\left(\frac{\pi}{2n}, x_2\right) = \frac{\sinh(nx_2)}{n^2} \to \infty$$

and

$$\int_{-\pi}^{\pi} u\left(\frac{\pi}{2n}, x_2\right)^2 dx_1 = \pi\left(\frac{\sinh(nx_2)}{n^2}\right)^2 \to \infty$$

as $n \to \infty$.

Thus the Cauchy problem is ill-posed in the sense that the solution does not depend continuously on the initial data (at least not in these topologies). The example can be modified so that arbitrarily many derivatives of the initial data also tend to 0, by changing $\sin(nx_1)/n$ to $\sin(nx_1)/n^k$, with k as large as wanted.

8. Using the solution formula

$$u(x,t) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} g(y)\,dy, \quad t \neq 0,$$

(see Section 4.3), we obtain

$$|u(x,t)| \leq \frac{1}{(4\pi|t|)^{n/2}} \int_{\mathbb{R}^n} |g(y)|\,dy, \quad t \neq 0.$$

Hence

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{n/2}} \|g\|_{L^1(\mathbb{R}^n)} \to 0$$

as $|t| \to \infty$. Note however, that the L^2 norm of the solution is conserved (see Section 4.3).