1 a) Let X be any set, $\mathcal{A} = \mathcal{P}(X)$ and μ = counting measure. Prove that

$$\mathcal{L}^p(X, \mathcal{A}, \mu) = L^p(X, \mathcal{A}, \mu)$$

This space is usually called $\ell^p(X)$, and we realize the functions as sequences $(a_k)_{k \in X}$. Whether $a_k \in \mathbb{C}$ or $\mathbb{R}$ is often implicit. If $X = \{1, 2, 3\}$, show that $\ell^2(X)$ "=" $\mathbb{R}^3$ and that $\|x-y\|_2$ is the "Euklidean distance", given by Pythagoras theorem. (1b comes later)

2 Given any X, show that if $p_1 > p_2$ then $\|f\|_{p_1} \leq \|f\|_{p_2}$ for $f \in \ell^{p_1}(X) \cap \ell^{p_2}(X)$. Conclude that $\ell^{p_2}(X) \subset \ell^{p_1}(X)$.

3 Given any $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$. Show that if $p_1 > p_2$ then $\|f\|_{p_2} \leq \|f\|_{p_1}$

4a Show that $L^{p_1}(X, \mathcal{A}, \mu) \subset L^{p_2}(X, \mathcal{A}, \mu)$ if μ is finite

4b Disprove the inclusion for a concrete choice of μ with $\mu(X)=\infty$

5 If q is the conjugate exponent of p, i.e. $\frac{1}{p}+\frac{1}{q}=1$. Given any $\langle g\rangle\in L^q(X,\mathcal{A},\mu)$ prove that

$$T_{\langle g\rangle}: L^p(X,\mathcal{A},\mu) \to \mathbb{C}$$

defined by

$$T_{\langle g\rangle}(\langle f\rangle)=\int fg\,d\mu$$

is a bounded linear operator with norm

$$\|T_{\langle g\rangle}\| \leq \|\langle g\rangle\|_{L^q(X,\mathcal{A},\mu)}.$$

($\langle g\rangle$ denotes the equivalence class of g)

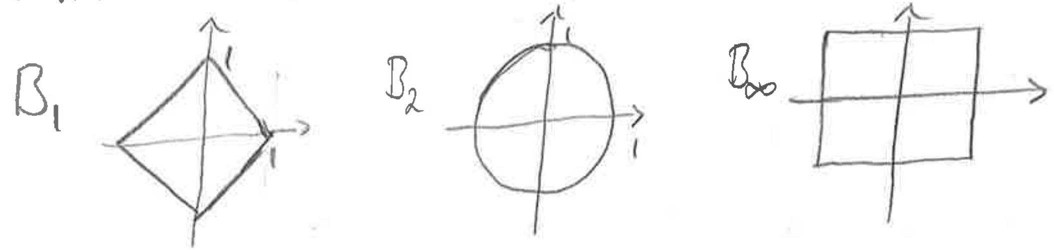
1b With $X=\{1,2\}$, paint the "unit ball"

$$B_p=\{x:\|x\|_p\leq 1\} \quad \text{for } p=1,2,\infty.$$

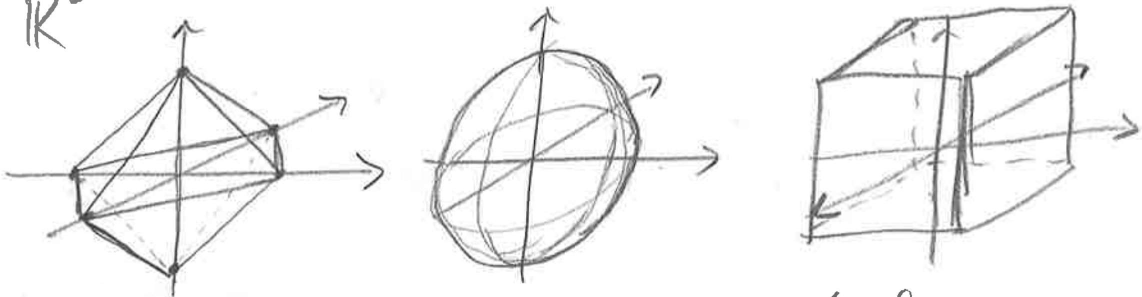
Try to visualize them for $X=\{1,2,3\}$. How many "flat faces" does B_1 have? Is it a rotated cube?

1a The equivalence class $[0]$ equals $\{0\}$ if and only if $\mu(E)=0 \Rightarrow E=\emptyset$, which is true for the counting measure.

1b In $\mathbb{R}^2$



In $\mathbb{R}^3$



8 faces

6 faces

(One face for each normal vector $(\pm 1, \pm 1, \pm 1)$)

2 By considering $f := \frac{f}{\|f\|_{p_2}}$, we can assume that $\|f\|_{p_2} = 1$. Then $|f(x)| \leq 1$ $\forall x \in X$ so

$$\int |f(x)|^{p_1} d\mu \leq \int |f(x)|^{p_2} d\mu = 1, \text{ since } p_1 > p_2.$$

Thus $\|f\|_{p_1} \leq 1$, as desired.

For the second part, note that $f \in L^{p_2}(X) \Rightarrow \|f\|_{p_2} < \infty \Rightarrow \|f\|_{p_1} < \infty \Rightarrow f \in L^{p_1}(X)$, ie $L^{p_2}(X) \subset L^{p_1}(X)$, as desired

3 Use Hölder's inequality with $p = \frac{p_1}{p_2}$ & $q = \frac{p}{p-1}$

$$\int |f|^{p_2} d\mu = \int (|f|^{p_2}) \cdot 1 \, d\mu \leq \| f \|_p \, \| 1 \|_q =$$

$$= \left(\int |f|^{p_1} d\mu \right)^{1/(p_1/p_2)} (\mu(X))^{1/q} = \left(\int |f|^{p_1} d\mu \right)^{p_2/p_1}$$

$\Rightarrow \| f \|_{p_2} \leq \| f \|_{p_1}$ as desired.

4a Set $\nu = \frac{1}{\mu(X)} \mu$. Then $f \in L^{p_1}(X, \mathcal{A}, \mu) \Rightarrow$

$f \in L^{p_1}(X, \mathcal{A}, \nu) \Rightarrow \| f \|_{L^{p_1}(\nu)} < \infty \Rightarrow \| f \|_{L^{p_2}(\nu)} < \infty \Rightarrow$

$f \in L^{p_2}(X, \mathcal{A}, \nu) \Rightarrow f \in L^{p_2}(X, \mathcal{A}, \mu)$

4b Take r with $\frac{1}{p_1} < r < \frac{1}{p_2}$ and consider

$f(x) = \chi_{[1,\infty)}(x) \cdot \frac{1}{x^r}$ with $X = \mathbb{R}$, $\mu = \lambda$.

Then $\int |f|^{p_1} d\lambda = \int_1^\infty \frac{1}{x^{rp_1}} dx < \infty$ since $rp_1 > 1$,

whereas $\int |f|^{p_2} d\lambda = \int_1^\infty \frac{1}{x^{rp_2}} dx = \infty$ since $rp_2 < 1$

Hence $L^{p_1}(X, \mathcal{B}(\mathbb{R}), \lambda) \not\subset L^{p_2}(X, \mathcal{B}(\mathbb{R}), \lambda)$

5 Sorry, it should be $\| T_{\langle g \rangle} \| \leq \| \langle g \rangle \|_{L^q(X, \mathcal{A}, \mu)}$ (not $L^p \ldots$).
This is immediate by Hölders inequality.