Exercise 1. A sequence $(a_k)_{k=1}^{\infty}$ converges absolutely if $\sum_{k=1}^{\infty} |a_k| < \infty$. In this case $\sum_k a_k$ exists and is independent of summation order, i.e. $\sum_k a_k = \lim_{j\to\infty} \sum_{k=1}^{j} a_{n_j}$ for any enumeration $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$. Show that this claim follows from DCT.

Exercise 2. Given $r<1$ and $\theta \in [-\pi, \pi)$, show that $\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$ "converges absolutely" and equals $\frac{1-r^2}{|1-re^{i\theta}|^2}$.

This is denoted the Poisson kernel. We write

$$P_r(\theta) = \frac{1-r^2}{2\pi |1-re^{i\theta}|^2}$$

a) Prove directly that

$$\lim_{r\to 1} P_r = 0$$

is true if the limit is interpreted λ-a.e. or λ-almost uniformly or in λ-measure.

b) With help of the theorems in the book, one of the above statements implies the other two. Which?

2c) Prove that

$$\lim_{r \to 1} P_r = 0$$

is false with respect to convergence in λ-mean. What should the limit be?

2d) Show that

$$\int_{[-\pi,\pi)} P_r(\theta)\, d\lambda(\theta) = 1$$

for all r.

2e) If $f \in C([-\pi,\pi))$ (continuous fcns on $[-\pi,\pi)$), show that

$$\lim_{r \to 1} \int f(\theta) P_r(\theta)\, d\lambda(\theta) = f(0) = \int f \, d\delta_0$$

(see 2g for definition of δ_0)

2f) Show that

$$\nu_r(E) = \int_E P_r(\theta)\, d\lambda(\theta)$$

defines a Borel measure on $[-\pi,\pi)$

2g) Let δ_0 be the Dirac measure on $[-\pi, \pi)$, i.e. $\delta_0(E) = \begin{cases} 1 & 0 \in E \\ 0 & 0 \notin E \end{cases}$

(The σ-algebra is understood to be $\mathcal{B}(\mathbb{R})$ restricted to $[-\pi, \pi)$)

Prove that

$$\lim_{r \to 1} P_r = \infty$$

δ_0-a.e., δ_0-almost uniformly & δ_0 in mean.

In the continuation course we shall prove the fundamental Riesz-representation theorem, which here implies that "the dual" (i.e. set of all bounded linear functionals) of $C([-\pi, \pi))$ "equals" all finite "signed" Borel measures.

The answer to the question in 2c is that the limit doesn't exist. However, thinking of P_r as measures as in 2f, 2e states that

$$\lim_{r \to 1} P_r = \delta_0 \text{ in the weak star topology.}$$

In other words, P_r is an "approximate identity."

1 Set $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and μ = counting measure.

A sequence $(a_k)_{k=1}^{\infty}$ then defines a function $a(k) = a_k$ on $\mathbb{N}$. If $a_k \geq 0$ $\forall k$, MCT gives

$$\int a(k)\, d\mu = \lim_{K\to\infty} \int a(k)\, \chi_{[1,K]}(k)\, d\mu$$

$$= \lim_{K\to\infty} \sum_{k=1}^{K} a_k = \sum_{k=1}^{\infty} a_k .$$

If $a_k \in \mathbb{C}$, $\forall k$, we have by Prop 2.6.4 that $\int a(k)\, d\mu$ exists iff $\int |a(k)|\, d\mu < \infty$. In this case, the DCT (with $g(k) = |a_k|$) gives

$$\int a(k)\, d\mu = \lim_{K\to\infty} \sum_{k=1}^{K} a_k = \sum_{k=1}^{\infty} a_k .$$

Similarly, if $(n_k)_{k=1}^{\infty}$ is an enumeration of $\mathbb{N}$, then $a(k)\chi_{\{n_1,\dots,n_K\}}(k)$ pointwise to a as $K\to\infty$ so DCT gives $\lim_{K\to\infty} \sum_{k=1}^{K} a_{n_k} = \lim_{K\to\infty} \int a \chi_{\{n_1,\dots,n_K\}}\, d\mu = \int a\, d\mu$

QED

2 As in the previous exercise we can interpret $\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$ as integrals on $\mathbb{Z}$. We have

$$\sum_{-\infty}^{\infty} r^{|k|} e^{ik\theta} = 1 + \sum_{k=1}^{\infty} r^{k} e^{ik\theta} + \sum_{k=1}^{\infty} r^{k} e^{-ik\theta}$$

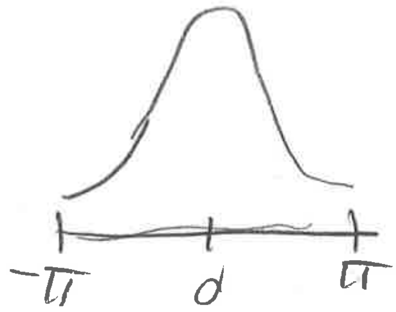
$$= 1 + 2\,\mathrm{Re} \lim_{K\to\infty} \sum_{k=1}^{K} (re^{i\theta})^k =$$

$$= 1 + 2\,\mathrm{Re} \lim_{K\to\infty} \frac{re^{i\theta} - (re^{i\theta})^K}{1 - re^{i\theta}} = 1 + 2\,\mathrm{Re} \frac{re^{i\theta}}{1 - re^{i\theta}}$$

$$= \mathrm{Re} \frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \mathrm{Re} \frac{(1 - re^{-i\theta})(1 + re^{i\theta})}{|1 - re^{i\theta}|^2}$$

$$= \mathrm{Re} \frac{1 - r^2 + 2ir\sin\theta}{|1 - re^{i\theta}|^2} = \frac{1 - r^2}{|1 - re^{i\theta}|^2}$$

$$= \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$ Note that $P_r(0) = \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r}$



2a) Note that P_r is even & decreasing for $\theta \in [0,\pi]$. For fix $\theta \neq 0$ we have

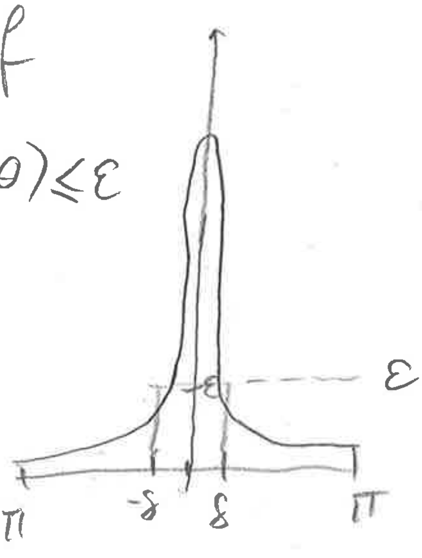
$$\lim_{r\to 1} |1-re^{i\theta}|^2 = |1-e^{i\theta}|^2 \neq 0, \text{ so}$$

$$\lim_{r\to 1} P_r(\theta) = \frac{\lim 1-r^2}{2\pi \lim |1-re^{i\theta}|^2} = 0$$

Thus P_r converges pointwise on $[-\pi,0)\cup(0,\pi)$, and hence λ-a.e.

Moreover, given any ε & $\delta > 0$ we have $\{\theta : P_r(\theta) > \varepsilon\} \subset [-\delta,\delta]$ if r is large enough that $P_r(\delta) \leq \varepsilon$

Thus $P_r \to 0$ uniformly on $[-\pi,-\delta)\cup(\delta,\pi)$



and hence $P_r \to 0$ almost uniformly.

Also $\limsup_{r\to 1} \lambda(\{\theta : |P_r(\theta) - 0| > \varepsilon\}) \leq$

$\leq \lambda([-\delta,\delta]) = 2\delta$, so $\limsup_{r\to 1} \lambda(\ldots) = 0$

and hence $P_r \to 0$ in measure

2b Since λ is finite on $[-\pi,\pi)$, the λ-a.e. convergence implies the other two by Prop

2d) Fix r and set $f_k(\theta)=\sum_{j=-k}^{k} r^{|j|}e^{ij\theta}$

Then $|f_k(\theta)| \leq \sum_{j=-\infty}^{\infty} r^{|j|} = \frac{1+r}{1-r}$. The constant function $g(\theta)=\frac{1+r}{1-r}$ is integrable on $X=[-\pi,\pi)$, so DCT gives

$$2\pi\int_{[-\pi,\pi)} P_r(\theta)\,d\lambda = \int_{[-\pi,\pi)} \lim_{k\to\infty} f_k(\theta)\,d\lambda = \lim_{k\to\infty}\int_{[-\pi,\pi)} f_k(\theta)\,d\lambda$$

$$= \lim_{k\to\infty}\sum_{j=-k}^{k}\int_{[-\pi,\pi)} e^{ij\theta}\,d\lambda = \lim_{k\to\infty} 2\pi = 2\pi$$

2c) By the above we have

$$\int_{[-\pi,\pi)} |P_r(\theta)-0|\,d\lambda = \int_{[-\pi,\pi)} P_r(\theta)\,d\lambda = \underline{1}$$

so $P_r \not\to 0$ in mean.

2e With ε, δ as in 2a) we have

$$\int_{[-\pi,\pi)} f P_r \, d\lambda = \int (f - f(0)) P_r \, d\lambda + \int f(0) P_r \, d\lambda.$$

Since $\int f(0) P_r \, d\lambda = f(0)$ and

$$\left| \int f - f(0) P_r \, d\lambda \right| \leq \int |f - f(0)| P_r \, d\lambda$$

$$\leq \int_{\{\pi > |\theta| > \delta\}} |f - f(0)| P_r \, d\lambda + \int_{\{|\theta| < \delta\}} |f - f(0)| P_r \, d\lambda$$

$$\leq 2\|f\|_\infty \int_{\{\pi > |\theta| > \delta\}} \varepsilon \, d\lambda + \sup_{|\theta| < \delta} |f(\theta) - f(0)| \cdot$$

$$\cdot \int_{\{|\theta| < \delta\}} P_r \, d\lambda \leq 2\|f\|_\infty 2\pi \varepsilon + \sup_{|\theta| < \delta} |f(\theta) - f(0)|$$

Since f is continuous, this can be made arbitrarily small by choosing δ small, so

$\lim_{r \to 1} \int_{[-\pi,\pi)} f P_r \, d\lambda = f(0)$, as desired