Exercise 1 Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$, and let $\alpha > 0$.

a) Prove that $f_\alpha : [1,\infty) \to \mathbb{R}^+$ defined by

$$f_\alpha(x) = \left(\frac{1}{\lfloor x \rfloor}\right)^\alpha$$

is $\mathcal{B}(\mathbb{R})$-measurable

b) Calculate $\int_{\mathbb{R}^+} f_\alpha \, d\lambda$, i.e. write it as an infinite sum.
(Hint; Use MCT.)

c) For which α is $\int_{\mathbb{R}^+} f_\alpha \, d\lambda < \infty$?

(You may use that $\underbrace{\int_{[a,b]} f \, d\lambda}_{\text{Lebesgue}} = \underbrace{\int_a^b f(x)\,dx}_{\text{Riemann}}$

for continuous f.)

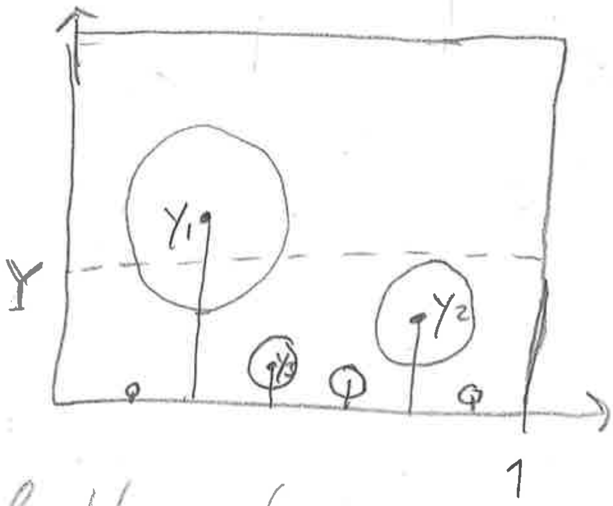
Exercise 2 Consider the square $[0,1]^2$

This is a cheese with holes.

Each hole sits at height y_k

with radius $y_k/2$.

Suppose that $\sum_{k=1}^{\infty} y_k < \infty$



Consider the lower part of the cheese $[0,1]\times[0,Y]$, where $0<Y<1$. Prove that the density d_Y of the cheese in this part approaches 1 as $Y \to 0$. Hints;

a) Prove $d_Y = \dfrac{\text{Area of cheese in } [0,1]\times[0,Y]}{1\cdot Y} \geq 1 - \dfrac{\frac{\pi}{4}\sum_{y_k\leq Y} y_k^2}{Y}$

b) Conclude that you need to show

$$\lim_{Y\to 0} \sum_{y_k \leq Y} y_k^2/Y = 0 \qquad (1)$$

c) Let μ be the measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ defined by $\mu(\{k\}) = y_k$, suppose that y_k is decreasing and set $f_k : \mathbb{N} \to \mathbb{R}^+$ via $f_k(j) = \begin{cases} y_j/y_k & j \geq k \\ 0 & j<k \end{cases}$. Apply DCT to get (1).

Exercise 2' Given $\alpha < 2$ find $y_k > 0$ s.t. $\sum_{k=1}^{\infty} y_k < \infty$ but

$$\lim_{\varphi \to 0} \frac{\sum_{y_k < \varphi} y_k^\alpha}{\varphi} = \infty$$

Exercise 3 Extend DCT as follows;

If $\varphi_\alpha \in L^1(X, \mathcal{A}, \mu)$ $\forall \alpha \in [0,1)$ is such that $\alpha \mapsto \varphi_\alpha(x)$ is continuous $\forall x \in X$, and $\exists\, g \in L^1(X, \mathcal{A}, \mu)$ with

$$|\varphi_\alpha(x)| \leq g(x) \quad \forall \alpha \ \forall x,$$

then

$$\lim_{\alpha \to 0} \int \varphi_\alpha \, d\mu = \int \varphi_0 \, d\mu$$

Hint Argue by contradiction to extract a sequence that contradicts DCT.

Exercise 4. Calculate

$$\int \frac{t}{1+(xt)^2}\, d\lambda(x) \qquad (t>0)$$

Calculate $\lim_{t\to 0} \frac{t}{1+(xt)^2}$

With suitable choice of $(X, \mathcal{A}, \mu)$ & φ_α, show that $|\varphi_\alpha(x)| \leq g(x)$ can not be omitted from Exercise 3.

Exercise 5 Let $f: X\times\mathbb{R}\to\mathbb{R}$ be such that $\partial_t f(x,t)$ exists $\forall x\in X$. Assume $\exists g\in L^1(X,\mathcal{A},\mu)$ s.t. $|\partial_t f(x,t)| \leq g(x)$ $\forall\, 0<t<1$. Set $F(t) = \int f(x,t)\, d\mu(x)$ and assume that $F(t)$ is well-defined (i.e. $f(\cdot,t)\in L^1(X,\mathcal{A},\mu)\ \forall t$). Prove that $F'(t)$ exists $\forall\ 0<t<1$ and that

$$F'(t) = \int \partial_t f(x,t)\, d\mu(x)$$

Exercise 6. Let $\gamma : [0,1] \to \mathbb{C}$ be injective, $\gamma(0) = \gamma(1)$ & continuously differentiable. Set $\Gamma = \gamma([0,1])$. Given a continuous function φ on Γ, we define $f : \mathbb{C} \setminus \Gamma \to \mathbb{C}$ by

$$f(z) = \int_\Gamma \frac{\varphi(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i} = \int_{[0,1]} \frac{\varphi(\gamma(t))}{\gamma(t) - z} \frac{\gamma'(t)}{2\pi i} \, d\lambda(t)$$

Write $z = x + iy$. Show that

$$\partial_x f(z) = \int_\Gamma \frac{\varphi(\zeta)}{(\zeta - z)^2} \frac{d\zeta}{2\pi i}.$$

Exercise 7. A function is called analytic if it satisfies Cauchy-Riemann's equation

$$\partial_x f = -i \partial_y f$$

Is f analytic?

Ansers I

1a) $f_\alpha^{-1}([t,\infty))$ is a halfaxis ...

1b) Set $f_k = f_\alpha \chi_{[1,k]}$. Then

$$\int f_\alpha \, d\lambda = \lim_{k\to\infty} \int f_k \, d\lambda = \lim_{k\to\infty} \sum_{j=1}^{k-1} \frac{1}{j^\alpha} = \sum_{j=1}^{\infty} \frac{1}{j^\alpha}$$

c) The hint is misleading (sorry)...

I just wanted you to compare

$\sum_{j=1}^{\infty} \frac{1}{j^\alpha}$ with $\int_1^\infty \frac{1}{x^\alpha} dx$, as in basic analysis. Answer $\alpha > 1$.

2a) $\frac{\pi}{4} \sum_{y_k \le Y} y_k^2$ is the area of all holes with center below Y...

2c) We have $|f_k| \le 1$ and $\int 1 \, d\mu < \infty$.

For $y_k \le Y < y_{k-1}$ we get by MCT

$$\sum_{y_j \le Y} y_j^2 / Y \le \sum_{j=k}^{\infty} \frac{y_j^2}{y_k} = \int f_k \, d\mu$$ and by

DCT we have $\lim_{k\to\infty} \int f_k \, d\mu = \int \lim_{k\to\infty} f_k \, d\mu = \int 0 \, d\mu = 0.$

2) Suppose $\alpha > 1$.
Pick $\beta > 1$ and $y_k = \frac{1}{k^\beta}$. We need to show

$$\lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{y_j^\alpha}{y_k} = \lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{k^\beta}{j^{\alpha\beta}} = \infty$$

As in Exercise 1 this is true iff

$$\lim_{k\to\infty} \int_k^\infty \frac{k^\beta}{x^{\alpha\beta}}\,dx = \lim_{k\to\infty} \left[\frac{k^\beta x^{1-\alpha\beta}}{1-\alpha\beta} \right]_k^\infty = \infty$$

If $\alpha\beta > 1$ the bracket evaluates to

$$\frac{k^{\beta+1-\alpha\beta}}{\alpha\beta - 1}$$

and we want to have

$$\beta + 1 - \alpha\beta < 0 \iff \beta < \frac{1}{\alpha - 1}.$$

Since $\alpha < 2$ such a $\beta > 1$ can be chosen.

Finally, if $\alpha < 1$ we can pick $\tilde{\alpha} > 1$ and y_k as above. Then

$$\lim_{k\to\infty} \frac{\sum_{j=k}^{\infty} y_j^\alpha}{y_k} \geq \lim_{k\to\infty} \frac{\sum_{j=k}^{\infty} y_j^{\tilde{\alpha}}}{y_k} = \infty$$

3 Do as the hint says.

4 $\int \frac{t}{1+(xt)^2} d\lambda(x) = \Big[\arctan(tx)\Big]_{-\infty}^{\infty} = \pi$

and $\lim_{t\to 0} \frac{t}{1+(xt)^2} = 0$ so with

$X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, $\mu = \lambda$ & $\varphi_\alpha(x) = \frac{\alpha}{1+(\alpha x)^2}$

we have $\lim_{\alpha\to 0} \int \varphi_\alpha d\lambda = \pi$ but $\int \varphi_0 d\lambda = 0$

5 $F(t+h) - F(t) = \int f(x,t+h) - f(x,t)\, d\mu(x)$

$= \int \partial_t f(x, t+\theta_{x,h} h) \cdot h \, d\mu(x)$

where $0 < \theta_{x,h} < 1$, (by some consequence of the mean value theorem). Setting

$\varphi_h = (f(x,t+h) - f(x,t))/h$ we thus have

$|\varphi_h(x)| \leq \partial_t f(x, t+\theta_{x,h} h) \leq g(x)$ and

$\lim_{h\to 0} \frac{F(t+h) - F(t)}{h} = \lim_{h\to 0} \int \varphi_h(x) d\mu = \int \lim_{h\to 0} \varphi_h(x)\, d\mu$

$= \int \partial_t f(x,t)\, d\mu(x)$ by Exercise 4.

sorry, I forgot to write $z \notin \Gamma$...

6 Apply 5 with (sorry for the switch of x/t...)

$$f(t,x) = \mathrm{Re}\, \frac{\varphi(\gamma(t))}{\gamma(t)-(x+iy)} \frac{\gamma'(t)}{2\pi i}$$

$X = [0,1]$, $\mu = \lambda$ (on $[0,1]$) and

$$g(t) = \sup_{t \in [0,1]} \left| \frac{\varphi(\gamma(t))\, \gamma'(t)}{(\gamma(t)-(x+iy))\, 2\pi} \right|$$

Do the same with the imaginary part, to deduce the formula. (Oups, $\varphi(z)$ should be $\varphi(\zeta)$)

7 Yes, $\partial_y f(z) = i \int_\Gamma \frac{\varphi(\zeta)}{(\zeta - z)^2} \frac{d\zeta}{2\pi i}$,

which is seen as in 6...