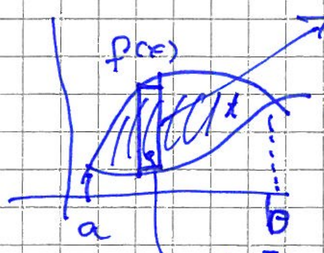
LV6-1

# Kap 14:

① Areor: $f(x) \geq g(x)$ i $[a,b]$

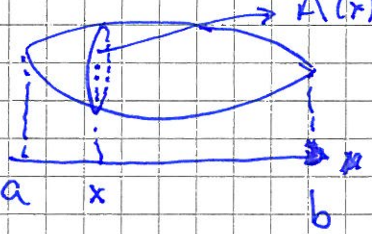



$$A = \int_a^b \left(f(x) - g(x)\right) dx$$

$A = [f(x_i) - g(x_i)](x_i - x_{i-1})$.

Kan tänkas som Riemann summor

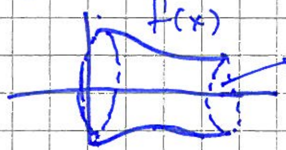
② Volym




$$V = \int_a^b A(x)\, dx$$

Kan tänkas som små Riemanncylinder med basarea $A(x)$ och höjd $dx$.

Rotationsvolym.

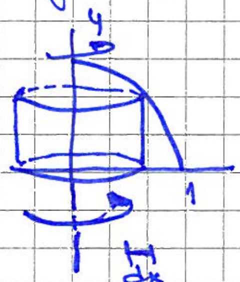



$A(x) = \pi f^2(x)$

$$V = \int_a^b \pi f^2(x)\, dx$$

(Rotation kring x-axeln).

Rotationsvolym kring y-axeln: $y = 1 - x^2$ $0 \leq x \leq 1$ roteras kring y-axeln.




Vi kan tänka oss en Riemannsumma av rörbitar dvs. cylinderskal

Varje cylinderrör har volym.

$2\pi x\, f(x)\, dx$

$$V = \int_{x_0}^{x_1} 2\pi x\, f(x)\, dx \quad \text{i detta fall} \quad V = \int_0^1 2\pi x (1 - x^2)\, dx = \pi/2$$

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Massa och tyngdpunkt: Om vi känner till $\rho(x)$ för en viss $dV$ kan vi anta $M = \int_K dm = \int_K \rho(x)\, dV$

Tyngdpunkt: $\boxed{X_{cm} = \frac{1}{M} \int_K \rho(x)\, x\, dV}$

Ex: En cylindrisk behållare är symmetriskt packade med $\rho(h) = 500 - 100h$ kg/m³ $0 \leq h \leq 4$ och $r = 1$

$dV = \pi\, dh$, $M = \int_0^4 \pi (500 - 100h)\, dh = \left[\pi(500h - 50h^2)\right]_0^4 = 1200\pi$.

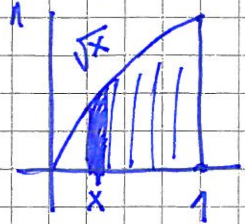
$h_{cm} = \frac{1}{1200\pi} \pi \int_0^4 h(500 - 100h)\, dh = \frac{1}{1200} \left[\frac{500h^2}{2} - \frac{100h^3}{3}\right]_0^4 \approx \frac{1867}{1200}$.

Tyngdpunkten för homogena 2-d skivor.

homogen: $\rho = 1$ $\therefore$ $M = \int_K 1 \cdot dV$ ; $x_{cm} = \int_K x\, dV$ ; $y_{cm} = \int_K y\, dV$

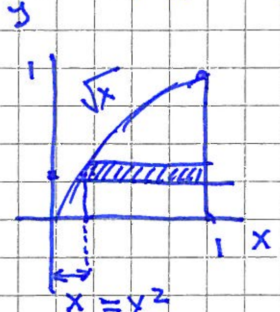
man måste ställa upp rätt integral.

Ex:




Massa: $\int_0^1 1 \cdot \sqrt{x}\, dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = 2/3$

$x_{cm}: \frac{1}{M} \int_0^1 x \sqrt{x}\, dx = \left[\frac{2}{5M} x^{5/2}\right]_0^1 = 2/5M = 3/5$




$y_{cm}: \frac{1}{M} \int_0^1 y\, (1 - y^2)\, dy = \left(\frac{y^2}{2} - \frac{y^4}{4}\right) \frac{1}{M} = \frac{1}{4M} = 3/8.$

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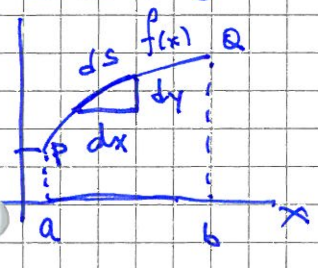
Kurvlängd




• kurvor som funktioner: $y = f(x)$, $a \leq x \leq b$

$$L_\Gamma = \int_P^Q ds = \int_a^b \sqrt{dx^2 + dy^2} = \boxed{\int_a^b \sqrt{1 + y'^2}\, dx}$$

• kurvor i parameter form: $\Gamma: \begin{cases} x(t) \\ y(t) \end{cases} \quad 0 \leq t \leq T$

Ex: cirkel: $\Gamma: \begin{cases} \cos t \\ \sin t \end{cases} \quad 0 \leq t \leq 2\pi$ $\quad \boxed{L_c = \int_0^{2\pi} \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt}$

• Kurvor i polärform: $\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan y/x \end{cases}$ där $\theta$ är parametern

$\therefore$ $\left.\begin{aligned} x(\theta) &= r(\theta) \cos\theta \\ y(\theta) &= r(\theta) \sin\theta \end{aligned}\right\}$ $x'^2 + y'^2 = (r'\cos - r\sin)^2 + (r'\sin + r\cos)^2 =$
$r^2 + r'^2$

$$L = \int_{\theta_0}^{\theta_1} \sqrt{r^2(\theta) + r'^2(\theta)}\, d\theta$$

# Kurvlängd exempel

LV6-3

14.29 $y = \ln(1-x^2)$, $0 \le x \le 1/2$ $\quad y' = \dfrac{-2x}{1-x^2}$

$$1+y'^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$$

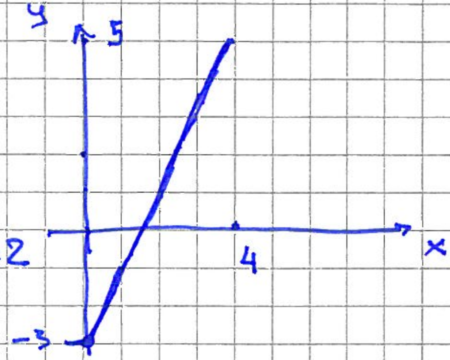
$$L = \int_0^{1/2} \sqrt{1+y'^2}\,dx = \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx = \int_0^{1/2} \left[-1 + \frac{2}{(1+x)(1-x)}\right] dx =$$

$$\frac{2}{(1+x)(1-x)} = \frac{1}{1-x} + \frac{1}{1+x} \quad \therefore \quad L = \left[-x + \ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2} =$$

$$= \boxed{\ln 3 - 1/2}$$

14.20 b. Bestäm längden av kurvan

$$\begin{cases} x(t) = 2t^2 - 4 \\ y(t) = 4t^2 - 11 \end{cases} \qquad \sqrt{2} \le t \le (\sqrt{2})^2 = 2$$




$(y = 2x - 3)$

$$\left.\begin{aligned} \dot{x} &= 4t \\ \dot{y} &= 8t \end{aligned}\right\} \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{16t^2+64t^2} = \sqrt{80}\,t$$

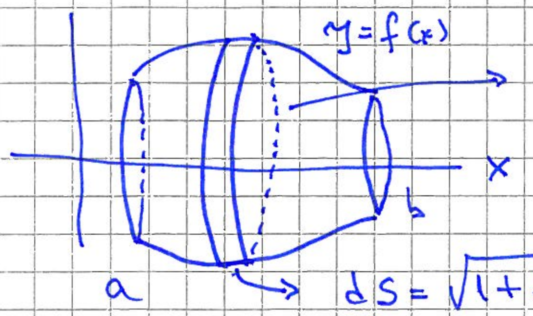
$$L = \int_{\sqrt{2}}^{2} \sqrt{80}\,t\,dt = \sqrt{80}\left[\frac{t^2}{2}\right]_{\sqrt{2}}^{2} = 2\sqrt{5}(4-2) = $$
$$= 4\sqrt{5}$$

För hand: $L = \sqrt{4^2+8^2} = \sqrt{80} = 4\sqrt{5}$

14.32 b: $\begin{cases} x = e^{-\theta/6}\cos\theta \\ y = e^{-\theta/6}\sin\theta \end{cases}$ ; $\theta \ge 0$ $\qquad r = e^{-\theta/6}$, $\quad r' = -1/6\, e^{-\theta/6}$

$$\sqrt{r^2+r'^2} = e^{-\theta/6}\sqrt{\frac{37}{36}} \qquad L = \sqrt{37}\int_0^{\infty} \frac{1}{6} e^{-\theta/6}\,d\theta = \sqrt{37}.$$

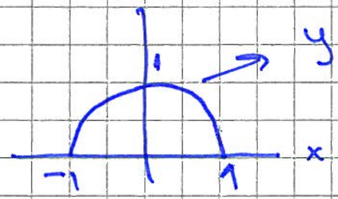
## Area av rotationsyta:




→ omkrets: $2\pi f(x)$

$ds = \sqrt{1+f'^2}$

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'^2(x)}\,dx$$




$y = \sqrt{1-x^2}$ $\quad -1 \le x \le 1$

Ex: Area av en sfär:

$y(x)$ roterar kring x-axeln och bildar en sfär med $r = 1$.

$$A = \int_{-1}^{1} 2\pi \sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

$$y' = \frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2}}$$

$$\sqrt{1+y'^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$= 4\pi.$

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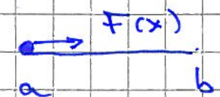
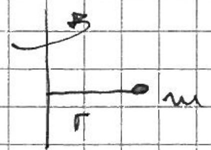
Mekaniskt arbete: $W = \int_a^b F(x)\cdot dx$




Tröghetsmoment: $J = \frac{1}{M}\int_k r^2 \rho\, dV$ där r är avstånd

till en axel i kroppen

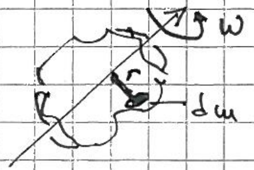
Ex:




kinetisk energi: $T = \frac{1}{2}mv^2 = \frac{1}{2}(mr^2)\omega^2$
för en ren rotation.

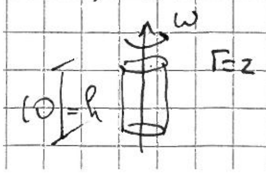
$mr^2 \to J$

För en hel kropp som roterar




$$T = \frac{1}{2}\int_k r^2 \rho(r)\,dV = \frac{1}{2}\int_k r^2\,dm$$

r är avstånd till rotationsaxeln.




$T = \frac{1}{2}\frac{h}{M}\int_0^2 2\pi r\, r^2\,dr\, \omega^2 = \frac{h}{2M}\, 2\pi \frac{2^4}{4}$. $M = 2\pi\, 2^2/2 \cdot h$ $\}\ J = 2$