Trigonometriska integraler:

$$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

Jämna potenser i $\cos x$, $\sin x$ kan hanteras med upprepad användning av "dubbla vinklar".

Ex $$\int \sin^4 x\,dx = \int (1-\cos^2 x)^2\,dx = \int (1 - 2\cos^2 x + \cos^4 x)\,dx$$

$$\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1 + 2\cos 2x + \cos^2 2x}{4}$$

och $$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\therefore \int \sin^4 x\,dx = x - 2\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right) + \left(\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right) + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + 1/32 \sin 4x + C$$

Udda potenser: $$\int \sin^5 x\,dx = \int \sin x\,(1-\cos^2 x)^2\,dx \qquad \begin{aligned} t &= \cos x \\ dt &= -\sin x\,dx. \end{aligned}$$

$$= -\int (1-t^2)^2\,dt = -\int (1 - 2t^2 + t^4)\,dt$$

$$= -\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C$$

För alla andra fall:

$$\begin{cases} \sin x = \dfrac{\sin 2\cdot x/2}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} = 2\,\dfrac{\sin x/2 \,\cos x/2}{\cos^2 x/2 + \sin^2 x/2} = \dfrac{2\tan x/2}{1+\tan^2 x/2} = \dfrac{2t}{1+t^2} \\[2ex] \cos x = \dfrac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 x/2 + \sin^2 x/2} = \dfrac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \dfrac{1-t^2}{1+t^2} \end{cases}$$

Substitutionen $t = \tan x/2$ och $x = 2\arctan t \Rightarrow$

$$dx = \frac{2\,dt}{1+t^2}$$

Detta omvandlar rationella fkn. av $\cos x$, $\sin x$ till rationella fc. av t.

Exempul

12.33 a) $\int \frac{1}{2+\sin x}\,dx = \left| \begin{array}{l} t = \tan x/2 \\ dx = \frac{2}{1+t^2}\,dt \\ \sin x = \frac{2t}{1+t^2} \end{array} \right| = \int \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$

$$I = \int \frac{1}{1+t+t^2}\,dt = \int \frac{1}{(t+1/2)^2+3/4}\,dt = \frac{4}{3}\int \frac{1}{\left[\frac{2}{\sqrt{3}}(t+\frac{1}{2})\right]^2+1}\,dt$$

$$z = \frac{2}{\sqrt{3}}(t+1/2) \Rightarrow dz = \frac{2}{\sqrt{3}}\,dt$$

$$I = \frac{4}{3}\frac{\sqrt{3}}{2}\int \frac{dz}{1+z^2} = \frac{2}{\sqrt{3}} \arctan\left\{\frac{2}{\sqrt{3}}\left(1/2 + \tan x/2\right)\right\} + C$$

Ex (S.299) $I = \int \frac{3+4\sin x}{(1+\cos x)^2}\,dx = \int \frac{3+4\frac{2t}{1+t^2}}{\left(1+\frac{1-t^2}{1+t^2}\right)^2} \cdot \frac{2}{1+t^2}\,dt$

$$= 2\int \frac{3(1+t^2)+8t}{(1+t^2+1-t^2)^2}\,dt = \int \frac{1}{2}\left(3+3t^2+8t\right)dt =$$

$$= \frac{3}{2}t + \frac{t^3}{2} + 2t^2 + C = \frac{1}{2}\left(3\tan\frac{x}{2} + \tan^3\frac{x}{2} + 4\tan^2\frac{x}{2}\right) + C$$