

Real-Time Systems
Formula Sheet
2011

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Aliasing

The fundamental alias for a frequency ω_1 is given by

$$\omega = |(\omega_1 + \omega_N) \bmod (\omega_s) - \omega_N|$$

ZOH-Sampling of a system with input time delay $\tau \leq h$

ZOH-sampling the continuous-time system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t - \tau)$$

gives

$$x(kh + h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh - h)$$

where

$$\Phi = e^{Ah}$$

$$\Gamma_0 = \int_0^{h-\tau} e^{As} ds B$$

$$\Gamma_1 = e^{A(h-\tau)} \int_0^{\tau} e^{As} ds B$$

Different ways of calculating Φ

$$\Phi = e^{Ah} = I + Ah + A^2 h^2 / 2 + \dots$$

$$\Phi = \mathcal{L}^{-1}(sI - A)^{-1}$$

Solution to system equation

The solution to the system equation

$$\begin{aligned} x(kh + h) &= \Phi x(kh) + \Gamma u(kh) \\ y(kh) &= Cx(kh) + Du(kh) \end{aligned}$$

is given by

$$Y(z) = C(zI - \Phi)^{-1} z x(0) + (C(zI - \Phi)^{-1} \Gamma + D) U(z)$$

Mapping of second-order poles

When a continuous-time system with the characteristic polynomial

$$s^2 + 2\zeta\omega_0 s + \omega_0^2$$

is ZOH-sampled the corresponding discrete-time characteristic polynomial is given by

$$z^2 + a_1 z + a_2$$

with

$$a_1 = -2e^{-\zeta\omega_0 h} \cos\left(\sqrt{1 - \zeta^2} \omega_0 h\right)$$

$$a_2 = e^{-2\zeta\omega_0 h}$$

Rules of thumb for sampling interval selection

Approximation of continuous-time design:

$$h\omega_c \approx 0.05 \text{ to } 0.15$$

where ω_c is the cross-over frequency (in radians per second) of the continuous-time system.

Discrete-time design:

$$\omega h = 0.1 \text{ to } 0.6$$

where ω is the desired natural frequency of the closed-loop system.

Some discrete-time functions and corresponding z -transforms

$f(k)$	$\mathcal{Z}(f(k))$
$\delta(k)$ (pulse)	1
1 (step)	$\frac{z}{z-1}$
k (ramp)	$\frac{z}{(z-1)^2}$
a^k	$\frac{z}{z-a}$

Some properties of the z -transform

1. Definition

$$F(z) = \sum_{k=0}^{\infty} f(kh)z^{-k}$$

2. Inversion

$$f(kh) = \frac{1}{2\pi i} \oint F(z)z^{k-1} dz$$

3. Linearity

$$\mathcal{Z}\{af + \beta g\} = a\mathcal{Z}f + \beta\mathcal{Z}g$$

4. Time shift

$$\mathcal{Z}\{q^{-n}f\} = z^{-n}F$$

$$\mathcal{Z}\{q^n f\} = z^n(F - F_1) \text{ where } F_1(z) = \sum_{j=0}^{n-1} f(jh)z^{-j}$$

5. Initial-value theorem

$$f(0) = \lim_{z \rightarrow \infty} F(z)$$

6. Final-value theorem

If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle, then

$$\lim_{k \rightarrow \infty} f(kh) = \lim_{z \rightarrow 1} (1 - z^{-1})F(z).$$

7. Convolution

$$\mathcal{Z}\{f * g\} = \mathcal{Z}\left\{\sum_{n=0}^k f(n)g(k-n)\right\} = (\mathcal{Z}f)(\mathcal{Z}g)$$

Zero-order hold sampling of a continuous-time system with transfer function
 $G(s)$

$G(s)$	$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$	
$\frac{1}{s}$	$\frac{h}{z-1}$	
$\frac{1}{s^2}$	$\frac{h^2(z+1)}{2(z-1)^2}$	
e^{-sh}	z^{-1}	
$\frac{a}{s+a}$	$\frac{1 - \exp(-ah)}{z - \exp(-ah)}$	
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah})$ $a_1 = -(1 + e^{-ah})$	$b_2 = \frac{1}{a}(1 - e^{-ah} - ah e^{-ah})$ $a_2 = e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $a_1 = -2e^{-ah}$	$b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_2 = e^{-2ah}$
$\frac{s}{(s+a)^2}$	$\frac{(z-1)he^{-ah}}{(z - e^{-ah})^2}$	
$\frac{ab}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{b(1 - e^{-ah}) - a(1 - e^{-bh})}{b - a}$ $b_2 = \frac{a(1 - e^{-bh})e^{-ah} - b(1 - e^{-ah})e^{-bh}}{b - a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$	
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha\left(\beta + \frac{\zeta\omega_0}{\omega}\gamma\right)$ $b_2 = \alpha^2 + \alpha\left(\frac{\zeta\omega_0}{\omega}\gamma - \beta\right)$ $a_1 = -2\alpha\beta$ $a_2 = \alpha^2$	$\omega = \omega_0\sqrt{1 - \zeta^2} \quad \zeta < 1$ $\alpha = e^{-\zeta\omega_0 h}$ $\beta = \cos(\omega h)$ $\gamma = \sin(\omega h)$

Observers

Observer (predictor case):

$$\hat{x}(k+1 | k) = \Phi \hat{x}(k | k-1) + \Gamma u(k) + K(y(k) - C \hat{x}(k | k-1))$$

Observer (filter case, with direct term):

$$\begin{aligned} \hat{x}(k | k) &= \Phi \hat{x}(k-1 | k-1) + \Gamma u(k-1) \\ &\quad + K[y(k) - C(\Phi \hat{x}(k-1 | k-1) + \Gamma u(k-1))] \\ &= (I - KC)(\Phi \hat{x}(k-1 | k-1) + \Gamma u(k-1)) + Ky(k) \end{aligned}$$

Approximation of continuous-time designs

Forward difference:

$$s' = \frac{z-1}{h}$$

Backward difference:

$$s' = \frac{z-1}{zh}$$

Tustin:

$$s' = \frac{2}{h} \cdot \frac{z-1}{z+1}$$

Tustin with prewarping:

$$s' = \frac{\omega_1}{\tan(\omega_1 h/2)} \cdot \frac{z-1}{z+1}$$

Approximation of continuous-time state feedback

$$\begin{aligned} u(t) &= Mu_c(t) - Lx(t) \\ u(kh) &= \tilde{M}u_c(kh) - \tilde{L}x(kh) \\ \tilde{L} &= L(I + (A - BL)h/2) \\ \tilde{M} &= (I - LBh/2)M \end{aligned}$$

EDF schedulability condition

$$U = \sum_{i=1}^n \frac{C_i}{T_i} \leq 1$$

RM schedulability conditions

$$\begin{aligned} \sum_{i=1}^n \frac{C_i}{T_i} &\leq n(2^{1/n} - 1) \\ \prod_{i=1}^{i=n} \left(\frac{C_i}{T_i} + 1\right) &\leq 2 \end{aligned}$$

DM schedulability condition

$$\sum_{i=1}^n \frac{C_i}{D_i} \leq n(2^{1/n} - 1)$$

Response time calculation

$$R_i = C_i + \sum_{\forall j \in hp(i)} \left\lceil \frac{R_i}{T_j} \right\rceil C_j$$

The Laplace transform

Operator lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Linearity
2	$F(s + a)$	$e^{-at} f(t)$	Damping
3	$e^{-as} F(s)$	$\begin{cases} f(t - a) & t - a > 0 \\ 0 & t - a < 0 \end{cases}$	Time delay
4	$\frac{1}{a} F\left(\frac{s}{a}\right) \quad (a > 0)$	$f(at)$	Scaling in t -domain
5	$F(as) \quad (a > 0)$	$\frac{1}{a} f\left(\frac{t}{a}\right)$	Scaling in s -domain
6	$F_1(s) F_2(s)$	$\int_0^t f_1(t - \tau) f_2(\tau) d\tau$	Convolution in t -domain
7	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\sigma) F_2(s - \sigma) d\sigma$	$f_1(t) f_2(t)$	Convolution in s -domain
8	$sF(s) - f(0)$	$f'(t)$	Differentiation in t -domain
9	$s^2 F(s) - sf(0) - f'(0)$	$f''(t)$	
10	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$	$f^{(n)}(t)$	
11	$\frac{d^n F(s)}{ds^n}$	$(-t)^n f(t)$	Differentiation in s -domain
12	$\frac{1}{s} F(s)$	$\int_0^t f(\tau) d\tau$	Integration in t -domain
13	$\int_s^\infty F(\sigma) d\sigma$	$\frac{f(t)}{t}$	Integration in s -domain
14	$\lim_{s \rightarrow 0} sF(s)$	$\lim_{t \rightarrow \infty} f(t)$	Final value theorem
15	$\lim_{s \rightarrow \infty} sF(s)$	$\lim_{t \rightarrow 0} f(t)$	Initial value theorem

Transform lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	1	$\delta(t)$	Dirac function
2	$\frac{1}{s}$	1	Step function
3	$\frac{1}{s^2}$	t	Ramp function
4	$\frac{1}{s^3}$	$\frac{1}{2} t^2$	Acceleration
5	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	
6	$\frac{1}{s+a}$	e^{-at}	
7	$\frac{1}{(s+a)^2}$	$t \cdot e^{-at}$	
8	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
9	$\frac{1}{1+as}$	$\frac{1}{a} e^{-t/a}$	
10	$\frac{a}{s^2+a^2}$	$\sin at$	
11	$\frac{a}{s^2-a^2}$	$\sinh at$	
12	$\frac{s}{s^2+a^2}$	$\cos at$	
13	$\frac{s}{s^2-a^2}$	$\cosh at$	
14	$\frac{1}{s(s+a)}$	$\frac{1}{a} (1 - e^{-at})$	
15	$\frac{1}{s(1+as)}$	$1 - e^{-t/a}$	
16	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-bt} - e^{-at}}{a-b}$	

Transform lexicon, continued

	Laplace transform $F(s)$	Time function $f(t)$
17	$\frac{s}{(s+a)(s+b)}$	$\frac{ae^{-at} - be^{-bt}}{a-b}$
18	$\frac{a}{(s+b)^2 + a^2}$	$e^{-bt} \sin at$
19	$\frac{s+b}{(s+b)^2 + a^2}$	$e^{-bt} \cos at$
20	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$\zeta = 0 \quad \frac{1}{\omega_0} \sin \omega_0 t$ $\zeta < 1 \quad \frac{1}{\omega_0 \sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin \left(\omega_0 \sqrt{1-\zeta^2} t \right)$ $\zeta = 1 \quad te^{-\omega_0 t}$ $\zeta > 1 \quad \frac{1}{\omega_0 \sqrt{\zeta^2-1}} e^{-\zeta\omega_0 t} \sinh \left(\omega_0 \sqrt{\zeta^2-1} t \right)$
21	$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	
	$0 \leq \tau \leq \pi :$	$\zeta < 1 \quad \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin \left(\omega_0 \sqrt{1-\zeta^2} t + \tau \right)$ $\tau = \arctan \frac{\omega_0 \sqrt{1-\zeta^2}}{-\zeta\omega_0}$
		$\zeta = 0 \quad \cos \omega_0 t$
		$\zeta = 1 \quad (1 - \omega_0 t)e^{-\omega_0 t}$
22	$\frac{a}{(s^2 + a^2)(s+b)}$	$\frac{1}{\sqrt{a^2 + b^2}} \left(\sin(at - \phi) + e^{-bt} \sin \phi \right)$ $\phi = \arctan \frac{a}{b}$

Laplace transform table, continued

	Laplace transform $F(s)$	Time function $f(t)$
23	$\frac{s}{(s^2 + a^2)(s + b)}$	$\frac{1}{\sqrt{a^2 + b^2}} (\cos(at - \phi) - e^{-bt} \cos \phi)$ $\phi = \arctan \frac{a}{b}$
24	$\frac{ab}{s(s + a)(s + b)}$	$1 + \frac{ae^{-bt} - be^{-at}}{b - a}$
25	$\frac{a^2}{s(s + a)^2}$	$1 - (1 + at)e^{-at}$
26	$\frac{a}{s^2(s + a)}$	$t - \frac{1}{a}(1 - e^{-at})$
27	$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{(b - c)e^{-at} + (c - a)e^{-bt} + (a - b)e^{-ct}}{(b - a)(c - a)(b - c)}$
28	$\frac{\omega_0^2}{s(s^2 + 2\zeta\omega_0s + \omega_0^2)}$	$0 < \zeta < 1$ $1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_0t} \sin(\omega_0\sqrt{1 - \zeta^2}t + \phi)$ $\phi = \arccos \zeta$ $\zeta = 0$ $1 - \cos \omega_0t$
29	$\frac{1}{(s + a)^{n+1}}$	$\frac{1}{n!} t^n e^{-at}$
30	$\frac{s}{(s + a)(s + b)(s + c)}$	$\frac{a(b - c)e^{-at} + b(c - a)e^{-bt} + c(a - b)e^{-ct}}{(b - a)(b - c)(a - c)}$
31	$\frac{as}{(s^2 + a^2)^2}$	$\frac{t}{2} \sin at$
32	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
33	$\frac{1}{\sqrt{s}} F(\sqrt{s})$	$\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\sigma^2/4t} f(\sigma) d\sigma$