FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 1: Introduction

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State-space models (autonomous systems)

- State vector: x → represent the memory that the dynamical system has of its past
- Input vector: u → typically control input applied to make the output to behave in a specific manner
- Output vector: y → variables of particular interest e.g. measurable or variables required to be controlled

- General: $f(x,u,y,\dot{x},\dot{u},\dot{y},\ldots)=0$
- Explicit: $\dot{x}=f(x,u), \quad y=h(x)$
- Affine in the control: $\dot{x}=f(x)+g(x)u, \quad y=h(x)$
- Linear time-invariant: $\dot{x}=Ax+Bu, \quad y=Cx$

Affine map
$f(x)=\underbrace{Ax}_{\text{Linear map}}+\underbrace{b}_{\text{"bias"}}$

$\dot{x}=\frac{dx}{dt}$

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Non-autonomous systems

- Autonomous forced system: $\dot{x}=f(x,u)$
- Control Input vector: $u=\gamma(t,x)$ → may contain functions of time e.g. feedforward terms estimated terms (adaptive control)
- Non-autonomous system: $\dot{x}=f(x,t)$

becomes unforced after substituting explicitely the controller

Always possible to transform to autonomous system

Introduce $x_{n+1}=$ time

$$\begin{aligned}\dot{x} &= f(x,x_{n+1})\\ \dot{x}_{n+1} &= 1\end{aligned}$$

if t is a result of adding an integrator in the control input then the system can be transformed to autonomous by introducing $x_{n+1}=\int_0^t x(\tau)d\tau$

$\dot{x}_{n+1}=x$

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Linear Systems

State space representation $\dot{x}=\frac{d}{dt}x(t)$

$$\begin{aligned}\dot{x}&=Ax+Bu\\ y&=Cx+Du\end{aligned}$$

Transfer function

$$H(s)=C(sI-A)^{-1}B+D$$
$$Y(s)=H(s)U(s)$$

State space models of systems are not unique

$$\dot{x}=\begin{bmatrix}0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & \cdots\\ \vdots & \vdots & \ddots & 1\\ -a_k & -a_{k-1} & \cdots & -a_1\end{bmatrix}x+\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}u \qquad x=\begin{bmatrix}y\\ \dot{y}\\ \vdots\\ y^{(k-1)}\end{bmatrix}$$

$$y^{(k)}+a_1y^{(k-1)}+\cdots+a_{k-1}\dot{y}+a_ky=u$$

- Controllable Canonical Form
- Observable Canonical Form

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Linear Systems

State space representation $\dot{x}=\frac{d}{dt}x(t)$

$$\begin{aligned}\dot{x}&=Ax+Bu\\ y&=Cx+Du\end{aligned}$$

Properties:
- Unique equilibrium if A is full-rank
- Regardless of the initial value, the equilibrium point is stable when the eigenvalues have negative real-parts
- Analytic solution: superposition of the natural modes of the system

Equilibrium $\dot{x}=0 \Rightarrow Ax=0$
if A full rank $x=0$

Lecture 02

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Linear Systems

State space representation $\dot{x}=\frac{d}{dt}x(t)$ $\qquad Y(s)=H(s)U(s)$

$$\begin{aligned}\dot{x}&=Ax+Bu\\ y&=Cx+Du\end{aligned}$$

Properties:
- Principle of superposition and scaling

$u \rightarrow \boxed{S} \rightarrow y=S(u)$

$$S(\alpha u_1+\beta u_2)=\alpha S(u_1)+\beta S(u_2)$$

- If the unforced system is asymptotically stable, the forced system is bounded-input bounded-output stable
- Sinusoidal input → Sinusoidal output at the same frequency

Properties:
- Unique equilibrium if A is full-rank
- Regardless of the initial value, the equilibrium point is stable when the eigenvalues have negative real-parts
- Analytic solution: superposition of the natural modes of the system

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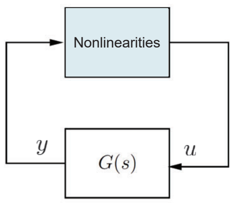
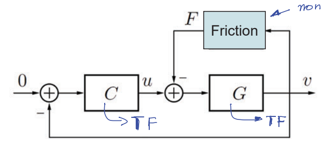
A standard form for analysis



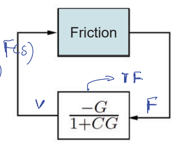
Example: closed loop with friction



nonlinear function e.g. $F = -d\,v|v|$

$F = -\text{sgn}(v)$ Coulomb friction

$\Longleftrightarrow$



$V(s) = G(s)\,(U(s) - F(s))$ (*)

$U(s) = C(s)\,(0 - V(s))$ (**)

Substituting (**) into (*)

$V(s) = -G(s)\,C(s)\,V(s) - G(s)\,F(s)$

$\Rightarrow (1 + G(s)\,C(s))\,V(s) = -G(s)\,F(s)$

$\Rightarrow$ (***)

(***) $V(s) = \dfrac{-G(s)}{1+G(s)C(s)} F(s)$

First order linear and nonlinear differential equations

- First order unforced systems described by differential equations

$\dot{x} = \dfrac{dx}{dt}$

$\underbrace{\dot{x} = -ax}_{\text{Linear } f(x) = -ax}$ $\quad \overbrace{\dot{x} = x^2 \quad \dot{x} = -\underbrace{|x|}_{\geq 0}x \quad \dot{x} = x(1-x)}^{\text{nonlinear}}$

- Damping – linear dynamic friction $\quad m\dot{v} + dv = u$
- Damping – nonlinear viscous friction (drag underwater vehicles) $\quad m\dot{v} + d|v|v = u$
- Population growth example $\quad \dot{N} = aN\left(1 - \dfrac{N}{M}\right)$

N → Population

M → Maximum size that can be reached

if $N = 0$ $\quad \dot{N} = 0$ remains 0

if $N = \epsilon > 0$ $\quad N$ will reach the maximum pop. M

Second order nonlinear equations

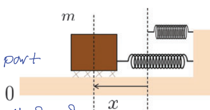
$\ddot{x} = \dfrac{d^2x}{dt^2}$

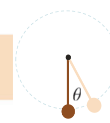
- Hardening spring $\quad \underbrace{m\ddot{x} + d\dot{x} + kx}_{\text{linear part}} + \underbrace{k\,ax^3}_{\text{nonlinear part}} = 0$
- Pendulum $\quad MR^2\ddot{\theta} + kR\dot{\theta} + MgR\sin\theta = 0$

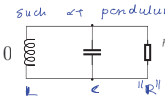
  In linear control systems $\sin\theta \sim \theta$ for small angles. This cannot hold for tasks such as pendulum swing-up
- Mechanical systems with friction $\quad m\ddot{x} + f(x, \dot{x}) = u$
- Circuit with negative resistance $\quad CL\ddot{v} + Lh'(v)\dot{v} + v = 0$

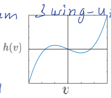
  ! many electronic components have non-linear characteristics
- Robot manipulators $\quad M\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + F_f(q, \dot{q}) = u$

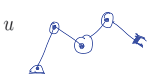
  M: positive definite but nonlinear inertia matrix











Transformation to first order systems & Equilibrium points

$\dot{x} = f(x)$

Assume $y^{(k)} = \frac{d^k y}{dt^k}$ highest derivative of y

Introduce $x = [\; y \quad \dot{y} \quad \ldots \quad y^{(k-1)} \;]^T$

Example: Pendulum

$MR\ddot{\theta} + k\dot{\theta} + Mg\sin\theta = 0$ $\quad$ 2nd order (DE)

$x = [\; \theta \quad \dot{\theta} \;]^T$ gives $\quad \ddot{\theta} = \frac{1}{MR}(-k\dot{\theta} - Mg\sin\theta)$

$\theta \to x_1$, $\dot{\theta} \to x_2$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1 \end{aligned}$$

$\dot{x}_1 = \dot{\theta} = x_2$

$\dot{x}_2 = \ddot{\theta} \overset{DE}{=} -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1$

Transformation to first order systems & Equilibrium points

- *The system can stay at equilibrium forever without moving*
- *Set all derivatives equal to zero!*

Assume $y^{(k)} = \frac{d^k y}{dt^k}$ highest derivative of y

Introduce $x = [\; y \quad \dot{y} \quad \ldots \quad y^{(k-1)} \;]^T$ ➡ Set $\dot{x} \equiv [\; \dot{y} \quad \ddot{y} \quad \ldots \quad y^{(k)} \;]^T = [\; 0 \quad 0 \quad \ldots \quad 0 \;]^T$

General: $f(x_0, u_0, y_0, 0, 0, 0, \ldots) = 0$

Explicit: $f(x_0, u_0) = 0$ $\Rightarrow$ nonlinear equation

Linear: $Ax_0 + Bu_0 = 0$ (has analytical solution(s)!) $\Rightarrow$ Linear Systems of Equations

if $u_0 = -Kx_0 \Rightarrow (A - BK)\,x_0 = 0$

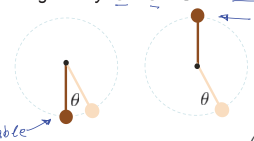
if u_0 is given as constant $\quad Ax_0 = -Bu_0 \Rightarrow x_0 = -A^{-1}Bu_0$

Multiple isolated equilibria

- Pendulum

$$MR\ddot{\theta} + k\dot{\theta} + Mg\sin\theta = 0$$

Equilibria given by $\ddot{\theta} = \dot{\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \quad \theta = \pm n\pi \qquad n \in \{0, 1, 2, 3, ..\}$



- Population growth

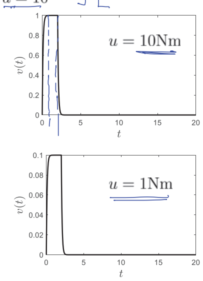
$$\dot{N} = aN\left(1 - \frac{N}{M}\right)$$

$N = 0 \qquad N = M$

Response to the input

$m\dot{v} + dv = u$ — pulse input

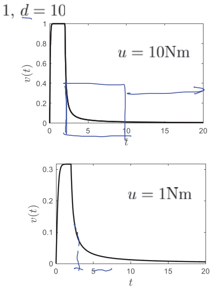
$m = 1,\ d = 10$



Linear System
Same response from $0 \to 1$ and $1 \to 0$

$m\dot{v} + d|v|v = u$

$m = 1,\ d = 10$



goes slower from $1 \to 0$
damping $\sim |v|$ smaller for low velocities

Finite escape time

Solution: $\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \Rightarrow \int \frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x} + c = t \Rightarrow \boxed{x = \frac{1}{c-t}}$ $\quad c = \frac{1}{x_0}$ $\quad x(t) = \frac{x_0}{1 - x_0 t}$

For $t = 0$: $-\frac{1}{x_0} + c = 0 \Rightarrow c = \frac{1}{x_0}$

Example: The differential equation

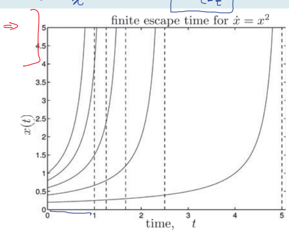
$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0$$

has solution

$$\boxed{x(t) = \frac{x_0}{1 - x_0 t}}, \qquad 0 \le t < \frac{1}{x_0}$$

Finite escape time: $t_f = \frac{1}{x_0}$

$$\lim_{t \to t_f} x(t) = \infty$$

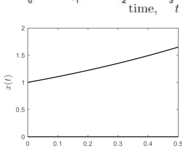


Compare with instability of linear systems

Example: $\quad \dot{x} = x, \quad x(0) = x_0$

Solution: $\quad x(t) = x_0 e^t \qquad \boxed{\lim_{t \to \infty} x(t) = \infty}$

the solution exists $\forall t$ but



Region of attraction

Region of attraction: The set of all initial conditions such that the solution converges to the equilibrium point

3 equilibrium points
$-x(1-x^2) = 0$
$x=0 \quad x=+1 \quad x=-1$

Example: The differential equation

$$\dot{x} = -x + x^3, \qquad x(0) = x_0$$

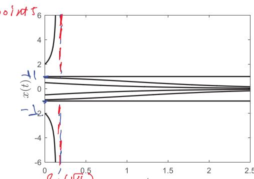
has solution

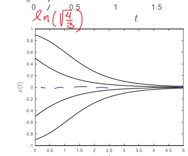
$$x(t) = \frac{x_0 e^{-t}}{\sqrt{1 - x_0^2 + x_0^2 e^{-2t}}}$$

- If $|x_0| \le 1$ the solution exists $\forall t \ge 0$
- If $|x_0| > 1$ the solution exists for:

$$0 \le t < \ln\sqrt{\frac{x_0^2}{x_0^2 - 1}}$$

finite escape time.

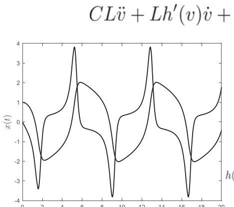


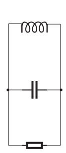


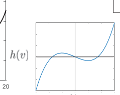
Limit Circles

- Circuit with negative resistance

$$CL\ddot{v} + Lh'(v)\dot{v} + v = 0$$







$$\boxed{\ddot{x} = -2(x^2 - 1)\dot{x} - x}$$

Non linear systems $\leadsto$ limit cycles
Amplitude independent of initial conditions

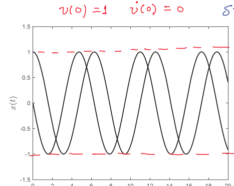
- Circuit with no resistance

Harmonic Oscillators
Linear Systems

$$CL\ddot{v} + v = 0$$

$v(0) = 1 \quad \dot{v}(0) = 0$

$R = 0$

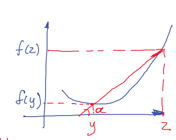


The amplitude of the oscillations depends on the initial conditions

$$\boxed{\ddot{x} = -x}$$

Lipschitz Continuity and existence and uniqueness of solutions

Lipschitz-continuous	$\|f(z) - f(y)\| \le L\|z - y\|$
Local	$\forall z, y \in \Omega_R$
Global	$\forall z, y \in \Re^n$



A straight line joining any two points of $f(x)$ cannot have a slope greater than L

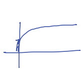
Infinite slope functions are not locally Lipschitz at the point of discontinuity. See e.g. $f(x) = \sqrt{x}$

Criterio for checking Lipschitz continuity

Check the continuity of $f(x)$ and the boundedness of first order partial derivatives $\frac{\partial f_i}{\partial x_j}$

Examples:

$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$



$f(x) = x^3 \qquad f'(x) = 3x^2 \to \infty$ for $x \to \infty$ — locally Lipschitz in a set around 0

$f(x) = \tanh x \qquad f'(x) = 1 - \tanh^2 x \to$ globally Lipschitz

Lipschitz Continuity and existence and uniqueness of solutions

Lipschitz-continuous $\|f(z) - f(y)\| \leq L\|z - y\|$ A solution of the diff. equation $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$ exists

Local $\forall z, y \in \Omega_R$ ➡ $x(t), \quad 0 \leq t < R/C_R, \quad C_R = \max_{x \in \Omega_R} \|f(x)\|$

Global $\forall z, y \in \Re^n$ ➡ $x(t), \quad t \geq 0.$

A solution of the diff. equation $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$ exists and is unique if

- $f(x)$ is locally Lipschitz $\forall x \in \Omega_R$
- **and** if it is known that every solution of the differential equation starting at a closed and bounded set $W \subset \Omega_R$ remains in it.

Lipschitz Continuity and existence and uniqueness of solutions

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Global $\forall z, y \in \Re^n$ ➡ $x(t), \quad t \geq 0.$

and

Local $\forall z, y \in \Omega_R \quad x(t) \in W \subset \Omega_R$ ➡ $x(t), \quad t \geq 0.$

Check the continuity of $f(x)$ and the boundedness of first order partial derivatives $\frac{\partial f_i}{\partial x_j}$

Examples: $f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$

$f(x) = x^3 \qquad f'(x) = 3x^2$

$f(x) = \tanh x \qquad f'(x) = 1 - \tanh^2 x$

Lipschitz Continuity and existence and uniqueness of solutions

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Global $\forall z, y \in \Re^n$ ➡ $x(t), \quad t \geq 0.$

and

Local $\forall z, y \in \Omega_R \quad x(t) \in W \subset \Omega_R$ ➡ $x(t), \quad t \geq 0.$

Check the continuity of $f(x)$ and the boundedness of first order partial derivatives $\frac{\partial f_i}{\partial x_j}$

Examples: $f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$

$f(x) = x^3 \qquad f'(x) = 3x^2$

$f(x) = \tanh x \qquad f'(x) = 1 - \tanh^2 x$

Existence of the solution, finite escape time, instability (Quiz)

$\boxed{\dot{x} = x^3}$

The only system that can have finite escape time

Bounded $\dot{x} = -\tanh x$ globally Lip.

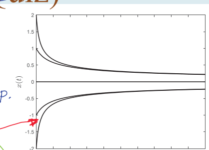
Bounded $\dot{x} = -x^3 = -x^2 \cdot x$ slower convergence

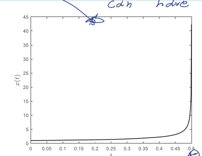
if $x_0 > 0$, $\dot{x} < 0$ decreasing to zero

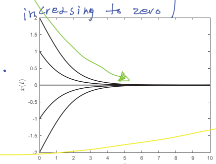
if $x_0 < 0$, $\dot{x} > 0$ increasing to zero

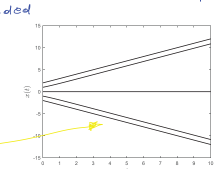
bounded

$\dot{x} = \tanh x$ globally Lip.









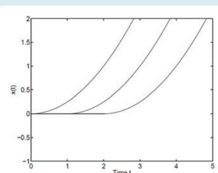
Uniqueness problems

Does the initial value problem have more than one solution?

If so, the differential equation cannot be used for prediction

Example: The equation $\dot{x} = \sqrt{x}, \quad x(0) = 0$ has many solutions:

$$x(t) = \begin{cases} (t - C)^2/4 & t > C \\ 0 & t \leq C \end{cases}$$



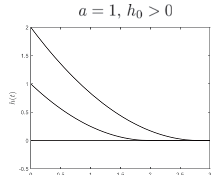
$a = 1, \, h_0 > 0$

Compare with water tank:

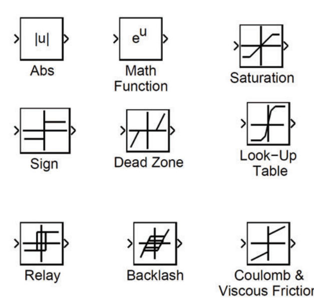
$$\dot{h} = -a\sqrt{h}, \qquad h: \text{height (water level)}$$

Change to backward-time: "If I see it empty, when was it full?"

$$h(t) = \begin{cases} (t - 2\sqrt{h_0})^2/4 & t < 2\sqrt{h_0} \\ 0 & t \geq 2\sqrt{h_0} \end{cases}$$



Some nonlinearities -simulink



Next Lecture(s)

- Phase plane analysis for 2nd order linear systems
- Linearization
- Stability definitions
- Simulation in Matlab

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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 2: Linearization and Phase plane analysis

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Outline

- Linearization around equilibrium
- Phase plane analysis of linear systems

Material

- Glad and Ljung: Chapter 13
- Khalil: Chapter 2.1–2.3
- Lecture notes

Linearization around an equilibrium point

- Linear systems with non-zero equilibrium points

Change of variables to move the origin to the equilibrium point

Example $\boxed{\dot{x} = Ax} + b$ $\dot{x} \equiv 0$

$Ax+b=0$ Equilibrium $x^\star = -A^{-1}b$

A full rank

New variable $\boxed{\tilde{x}} = x - x^\star$

$x = \tilde{x} + x^\star$

$\dot{\tilde{x}} = \dot{x} + \cancel{\dot{x}^\star}$ 0 constant

$\boxed{\dot{\tilde{x}} = A\tilde{x}}$

- Linear approximation of nonlinear systems (Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$)

$f^{(n)} = \frac{\partial^n f}{\partial x^n}$

$\dot{x} = f(x)$

$\dot{\tilde{x}} = \dot{x}$

$\dot{x} \equiv 0$ Equilibrium $f(x^\star) = 0$

$x = \tilde{x} + x^\star$

$\boxed{\dot{\tilde{x}} = f(\tilde{x} + x^\star)}$

New variable $\tilde{x} = x - x^\star$

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Linearization around an equilibrium point

- Linear approximation of nonlinear systems (Taylor expansion)

$\dot{\tilde{x}} = f(\tilde{x} + x^\star)$

$\boxed{f(x)} = f(\cancel{x}) + J_f(x^\star)\tilde{x} + \frac{1}{2}\tilde{x}^T H_f(x^\star)\tilde{x} + \cdots$

Jacobian Hessian

$\begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots x_n \\ f_n(x_1, x_2, \ldots, x_n \end{bmatrix}$

e.g. $f(x) = \begin{bmatrix} x_2 \\ f_2(x_1, x_2) \end{bmatrix}$

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$J_f = \begin{bmatrix} 0 & 1 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$

$\dot{\tilde{x}} = f(x^\star) + \frac{\partial f(\tilde{x}+x^\star)}{\partial \tilde{x}}\Big|_{\tilde{x}=0} \tilde{x} + \text{H.O.T.}$

$\tilde{x} = x - x^\star$

$\dot{\tilde{x}} = f(x^\star) + \frac{\partial f}{\partial x}(x^\star)\tilde{x} + \text{H.O.T.}$

equilibrium

$\boxed{f(x^\star) = 0}$

$\dot{\tilde{x}} = \underbrace{\frac{\partial f}{\partial x}(x^\star)}_{J_f}\tilde{x} + \text{H.O.T.}$

for small scalar $\tilde{x}$

$f(\tilde{x}) = 2\tilde{x} + \tilde{x}^2$

$\Rightarrow \tilde{x}^2 << \tilde{x}$

$f(\tilde{x}) \simeq 2\tilde{x}$

$\dot{\tilde{x}} \approx \frac{\partial f}{\partial x}(x^\star)\tilde{x}$

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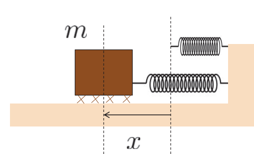
Example (nonlinear spring with external force)



- Differential Equation

$$m\ddot{x} + k_v\dot{x} + k_s x^3 = F$$

- State space representation

Position: $x_1 = x$ **Velocity:** $x_2 = \dot{x}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{k_s}{m}x_1^3 - \frac{k_v}{m}x_2 + \frac{F}{m}$

- State space representation (vector form) $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$

$$\dot{x} = f(x) \quad \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} f(x) = \begin{bmatrix} x_2 \\ -\frac{k_s}{m}x_1^3 - \frac{k_v}{m}x_2 + \frac{F}{m} \end{bmatrix}$$

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$$\dot{x}_1 = f_1(x) = x_2$$
$$\dot{x}_2 = f_2(x) = -\frac{k_s}{m}x_1^3 - \frac{k_v}{m}x_2 + \frac{F}{m}$$

Equilibrium $\dot{x}_1 \equiv 0$, $\dot{x}_2 \equiv 0$
$$\begin{cases} x_2 = 0 \\ k_s x_1^3 = F \Rightarrow x_1^3 = \frac{F}{k_s} \Rightarrow x_1 = \left(\frac{F}{k_s}\right)^{1/3} \end{cases}$$

Jacobian

$$J_f = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3k_s x_1^2}{m} & -\frac{k_v}{m} \end{bmatrix}$$

$$J_f(x^*) = J_f\Big(\underbrace{\left(\tfrac{F}{k_s}\right)^{1/3}}_{x_{1,eq}}, \underbrace{0}_{x_{2,eq}}\Big)$$

Jacobian calculated at equil.

$$J_f(x^*) = \frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} 0 & 1 \\ -\frac{3k_s}{m}\left(\frac{F}{k_s}\right)^{2/3} & -\frac{k_v}{m} \end{bmatrix}$$

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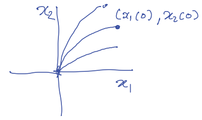
Phase plane analysis

- The phase plane method is the graphical study of second-order autonomous systems:

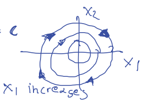
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$



- Phase plane has x_1 and x_2 as coordinates. $\Rightarrow x_1(t)$, $x_2(t)$
- Phase plane trajectory: a curve of the phase plane representing the solution for initial conditions $x_1(0),\ x_2(0)$ with time t varied from 0 to infinity
- Phase portrait: a family of phase plane trajectories from various initial conditions
- Example: $\ddot{y} + y = 0$ $\quad x_1 = y(t) = c\sin(t+\phi_0)$, $x_2 = \dot{y}(t) = c\cos(t+\phi_0)$

$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$

$x_1^2 + x_2^2 = c$



for $x_2 > 0$, $\dot{x}_1 > 0 \Rightarrow x_1$ increases

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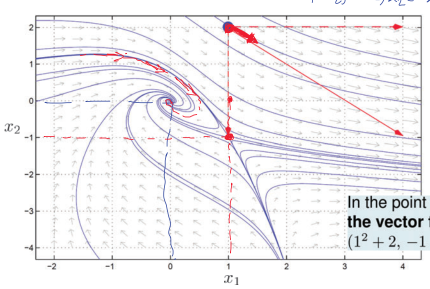
A first glimpse on phase portraits

$$\dot{x}_1 = f_1(x_1, x_2) = x_1^2 + x_2$$
$$\dot{x}_2 = f_2(x_1, x_2) = -x_1 - x_2$$

Eq. $x_1^2 + x_2 = 0$ (*) $\Rightarrow x_1^2 - x_1 = 0$; $-(x_1 + x_2) = 0 \Rightarrow x_2 = -x_1$ (**)

$x_1 = 0,\ x_2 = 0$; $x_1 = 1,\ x_2 = -1$



- The **vector field** $f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$ is tangent at point (x_1, x_2) because

$$\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

- The slope is indeterminate at equilibrium points aka singular points
- Don't forget the arrows!

In the point $(x_1, x_2) = (1, 2)$ **the vector field** is pointing in the direction $(1^2 + 2, -1 - 2) = (3, -3)$.

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Solution of Linear Systems of diff. eq.

State space representation: $\dot{x} = \boxed{A}x$ $\quad \dot{x} = ax \Rightarrow x(t) = x(0)e^{at}$ similar to scalar

Solution: $\boxed{x = e^{At}x(0)}$

Similarity transformation and change of variables: $\quad z = Wx$

$$\dot{x} = \boxed{A}x \Rightarrow \dot{x} = W^{-1}MWx \Rightarrow \boxed{\dot{z} = Mz}$$

Real distinct eigenvalues $\lambda_1,\ \lambda_2$	*One double eigenvalue* λ	*Complex Eigenvalues* $\sigma \pm j\omega$
$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$M = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$M = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$

Two real eigenvectors $\quad$ *Complex eigenvectors*

$W = [\ v_1 \quad v_2\ ]$ $\quad W = [\ v \quad R_{\frac{\pi}{2}}v\ ]$ $\quad W = [\ \Re(v_1) \quad \Im(v_1)\ ]$

Eigenvalues $\det(\lambda I - A) = 0$ $\quad$ Eigenvectors $Av = \lambda v$ $\quad$ Rotation matrix $R = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$ $\quad$ Real $\quad$ Imaginary

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Solution of Linear Systems of diff. eq.

State space representation after change of variables $z = Wx$: $\quad \dot{z} = Mz$

Solution for the new state: $\quad z(t) = e^{Mt}z(0)$

Real distinct eigenvalues $\quad e^{Mt} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t})$ $\quad W = [\ v_1 \quad v_2\ ]$

One double eigenvalue $\quad e^{Mt} = \mathrm{diag}(e^{\lambda t} + te^{\lambda t}, e^{\lambda t})$ $\quad W = [\ v \quad R_{\frac{\pi}{2}}v\ ]$

Complex Eigenvalues $\quad e^{Mt} = e^{\sigma t I}e^{\omega t S} = \boxed{e^{\sigma t}}R_{\omega t}$ $\quad W = [\ \Re(v_1) \quad -\Im(v_1)\ ]$

$R_{\omega t} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$

Solution of the original state: $\quad x(t) = W^{-1}z(t) = W^{-1}e^{Mt}z(0)$

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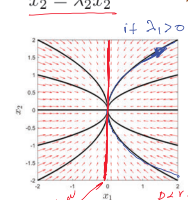
Two real eigenvalues

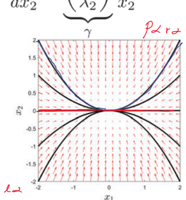
Direct elimination of time variable $\quad$ Separation of variables $\quad$ Integration

$$\begin{aligned}\dot{x}_1 &= \lambda_1 x_1 \\ \dot{x}_2 &= \lambda_2 x_2\end{aligned} \Rightarrow \frac{dx_1}{dx_2} = \underbrace{\left(\frac{\lambda_1}{\lambda_2}\right)}_{\gamma}\frac{x_1}{x_2} \Rightarrow \frac{dx_1}{x_1} = \gamma\frac{dx_2}{x_2} \Rightarrow \boxed{x_1 = Cx_2^{\gamma}}$$



$\gamma > 1$

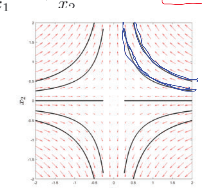
$\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0 \qquad 0 < \underbrace{\lambda_2}_{\text{slower}} < \underbrace{\lambda_1}_{\text{faster}}$



$0 < \gamma < 1$

$x_1 = Cx_2^{\gamma}$

$\Rightarrow x_2 = \left(\frac{1}{C}x_1\right)^{1/\gamma}$



$\gamma < 0$

$\lambda_2 < 0 < \lambda_1$

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Two real eigenvalues

$\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0$

v_1 $\quad$ v_2

$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Fast eigenvector

$x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ $\boxed{\text{for small } t}$

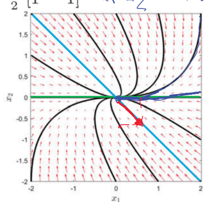
Slow eigenvector

$x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t

Example: $[X\ Y] = \mathrm{eigs}\ (A)$

$\dot{x} = \overbrace{\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}}^{A} x$ $\qquad (\lambda_1, \lambda_2) = (-1, -2)$

$v_2 = \frac{\sqrt{2}}{2}[1\ -1]^T$ ← -2 faster



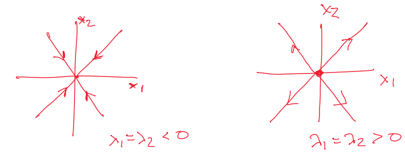
-1

$v_1 = [1\ 0]^T$ slower

Some comments

- What if $\lambda_1 = \lambda_2$? $\quad x_1 = c\, x_2$ 1

Star



- $\lambda_1 \lambda_2 = \det(A)$

  $\lambda_1 + \lambda_2 = \mathrm{Tr}(A)$

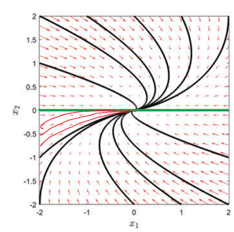
  $\lambda_{1,2} = \frac{1}{2}\left[\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}^2(A) - 4\det(A)}\right]$

One tangent mode

$\dot{x} = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} x, \qquad \mathrm{rank}\,(\lambda I - A) = 1$

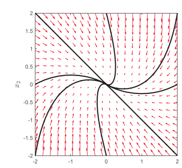
$x_1(t) = x_1(0)e^{\lambda t} + t x_2(0) e^{\lambda t}$

$x_2(t) = x_2(0) e^{\lambda t}$

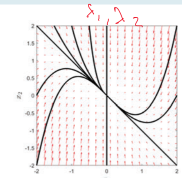


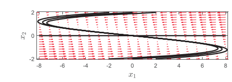
$v_1 = [1\ 0]^T$

Matching quiz 1



λ_1, λ_2





$\dot{x} = \begin{bmatrix}-0.5 & 0.5\\ -0.5 & -1.5\end{bmatrix} x \qquad \dot{x} = \begin{bmatrix}-1 & -10\\ 0 & -1\end{bmatrix} x \qquad \dot{x} \begin{bmatrix}-1 & 0\\ -2 & -3\end{bmatrix} x$

$\dot{x}_2 = -x_2$

Complex eigenvalues

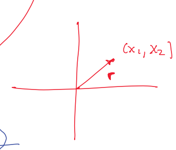
$\dot{x} = Ax \xrightarrow{z = W^{-1}x} \dot{z} = \begin{bmatrix}\sigma & -\omega\\ \omega & \sigma\end{bmatrix} z \xrightarrow{r = \sqrt{z_1^2 + z_2^2},\ \theta = \arctan z_2/z_1} \boxed{\begin{aligned}\dot{r} &= \sigma r\\ \dot{\theta} &= \omega\end{aligned}}$

$\sigma \pm j\omega$

$W = [\ \Re(v_1)\ \ \Im(v_1)\ ]$

$z_1 = r\cos\theta,\ z_2 = r\sin\theta$

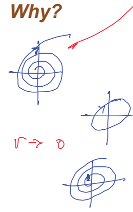
Why?



If $\sigma > 0$ Unstable focus $\qquad \dot{r} = 6r$

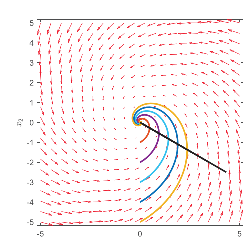
If $\sigma = 0$ Center $\qquad \dot{r} = 0$

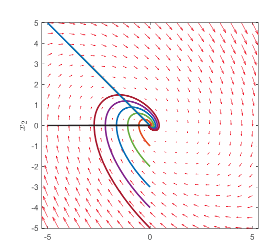
If $\sigma < 0$ Stable focus $\qquad \dot{r} = -2r \quad r \to 0$



Matching quiz 2

$\dot{x} = \frac{1}{2}\begin{bmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{bmatrix} x \qquad \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j \qquad \dot{x} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix} x$

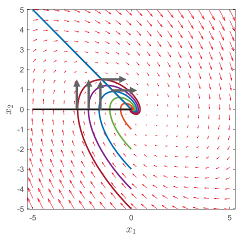


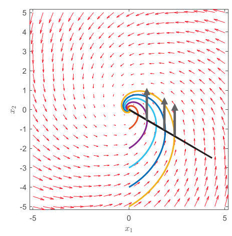


Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x \quad \xrightarrow{\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j} \quad \dot{z} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} z \quad \longrightarrow \quad \dot{r} = -\frac{1}{2} r, \quad \dot{\theta} = \frac{\sqrt{3}}{2}$$

- Stable focus $\sigma = -1/2 < 0$





How to draw phase portraits

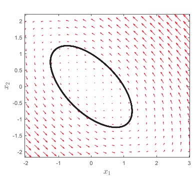
If done by hand then

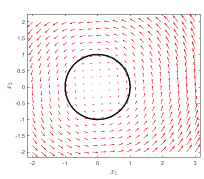
1. Find equilibria (also called singularities)
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Use that $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$.
4. Try to find possible limit cycles
5. Guess solutions

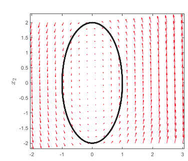
Matlab: PPTool and some other tools for Matlab is available on Canvas.

Matching quiz 3

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} x \qquad \dot{x} = \begin{bmatrix} -1.5 & -2.5 \\ 2.5 & 1.5 \end{bmatrix} x \qquad \dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x$$

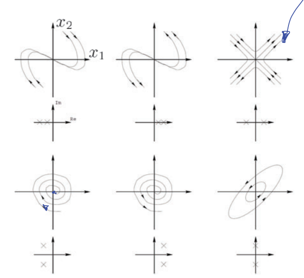






Summary of phase portraits and their equilibriums

Im$\lambda_i = 0$:	stable node $\lambda_1, \lambda_2 < 0$	unstable node $\lambda_1, \lambda_2 > 0$	saddle point $\lambda_1 < 0 < \lambda_2$
Im$\lambda_i \neq 0$:	Re$\lambda_i < 0$ stable focus	Re$\lambda_i > 0$ unstable focus	Re$\lambda_i = 0$ center point



Effect of perturbations

Perturbations in $A + \Delta$

- Structurally stable: the qualitative behavior remains the same under arbitrarily small perturbations in A

  Examples: a node(with distinct eigenvalues), a saddle or a focus

- A stable node with multiple eigenvalues could become a stable node or a stable focus under arbitrarily small perturbations in A

- A center is not structurally stable

$$\dot{z} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} z \quad \Rightarrow \quad \dot{z} = \begin{bmatrix} \delta & -\omega \\ \omega & \delta \end{bmatrix} z \qquad \delta \pm j\omega$$

Back to linearization

Theorem Assume

$$\dot{x} = f(x)$$

is linearized at $x^\star$ so that

$$\dot{\tilde{x}} = A\tilde{x} + g(x),$$

where

- $A = \frac{\partial f}{\partial x}(x^\star)$
- $g = f(x) - \frac{\partial f}{\partial x}(x^\star)\tilde{x} \in C^1$ and $\frac{\|g(x)\|}{\|\tilde{x}\|} \to 0$ as $\|\tilde{x}\| \to 0$

If $\dot{\tilde{x}} = A\tilde{x}$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

If the linearized system has a center, then the nonlinear system has either a center or a focus.

Back to linearization

Summary

- Linearization

$$\dot{x} = f(x) \Rightarrow \dot{\tilde{x}} = A\tilde{x} + g(x),$$

- Phase portraits of Linear systems
- Whether the behavior of the Linear system (outcome of linearization) can be inherited to the nonlinear system?

Lecture 3: Linearization around a trajectory, Limit cycles and Stability definitions

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Outline

- Linearization around trajectory (general case)
- Limit Cycles
- Definitions of Stability

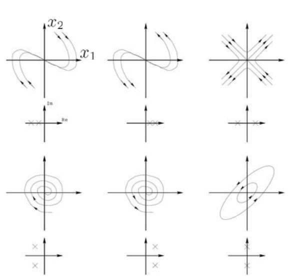
Material

- Glad& Ljung Ch. 11, 12.1, (Khalil Ch 2.3, part of 4.1, and 4.3)
- Lecture slides

Summary of phase portraits and their equilibriums

Im$\lambda_i = 0$:	stable node $\lambda_1, \lambda_2 < 0$	unstable node $\lambda_1, \lambda_2 > 0$	saddle point $\lambda_1 < 0 < \lambda_2$
Im$\lambda_i \neq 0$:	Re$\lambda_i < 0$ stable focus	Re$\lambda_i > 0$ unstable focus	Re$\lambda_i = 0$ center point

$\dot{x} = Ax$



Effect of perturbations

Perturbations in $A + \Delta$

- Structurally stable: the qualitative behavior remains the same under arbitrarily small perturbations in A

  Examples: a node(with distinct eigenvalues), a saddle or a focus

- A stable node with multiple eigenvalues could become a stable node or a stable focus under arbitrarily small perturbations in A

- A center is not structurally stable

$\Delta = \delta I$

$$\dot{z} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} z \quad \Rightarrow \quad \dot{z} = \begin{bmatrix} \delta & -\omega \\ \omega & \delta \end{bmatrix} z \qquad \boxed{\delta \pm j\omega}$$

Linearization around an equilibrium point

- Linear approximation of nonlinear systems (Taylor expansion)

$\dot{\tilde{x}} = f(\tilde{x} + x^\star)$ $\longrightarrow$ $\dot{\tilde{x}} = f(x^\star) + \frac{\partial f(\tilde{x}+x^\star)}{\partial \tilde{x}}\Big|_{\tilde{x}=0} \tilde{x} + \text{H.O.T.}$

$\tilde{x} = x - x^\star$

$$f(x) = f(x^\star) + J_f(x^\star)\tilde{x} + \frac{1}{2}\tilde{x}^T H_f(x^\star)\tilde{x} + \cdots$$

Jacobian Hessian

$\dot{\tilde{x}} = f(x^\star) + \frac{\partial f}{\partial x}(x^\star)\tilde{x} + \text{H.O.T.}$

$f(x^\star) = 0$

$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(x^\star)\tilde{x} + \text{H.O.T.}$

$\dot{\tilde{x}} \approx \frac{\partial f}{\partial x}(x^\star)\tilde{x}$

$\frac{\partial f}{\partial x}$

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$

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Predicting behaviors close to equilibrium

$\dot{x} = f(x)$

$\dot{\tilde{x}} = A\tilde{x} + g(x)$,

Linear approximation

$\tilde{x} = x - x^\star$ $A = \frac{\partial f}{\partial x}\Big|_{x=x^\star}$

$\dot{\tilde{x}} \simeq A\tilde{x}$ *Valid when* $\det(A) \neq 0$

$\lambda_1, \lambda_2 \neq 0$

where $A = \frac{\partial f}{\partial x}(x^\star)$
and $g = f(x) - \frac{\partial f}{\partial x}(x^\star)\tilde{x} \in C^1$ and $\frac{\|g(x)\|}{\|\tilde{x}\|} \to 0$ as $\|\tilde{x}\| \to 0$.

$\dot{x} = f(x)$ $\dot{\tilde{x}} \simeq A\tilde{x}$

???	Center
node	node
saddle	saddle
focus	focus
unstable	$\lambda_1, \lambda_2 > 0$
stable	$\lambda_1, \lambda_2 < 0$
???	$\lambda_i = 0$

$\rightarrow$ unstable

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Linearization around a trajectory

Idea: Make Taylor-expansion around a known solution $\{x^\star(t), u^\star(t)\}$ satisfying the differential equation:

$$\frac{dx^\star}{dt} = f(x^\star(t), u^\star(t))$$

$\dot{x} = f(x, u)$

Equilibrium points

$\frac{dx^\star}{dt} = 0$

$\tilde{x} = x - x^\star$

$$\dot{x}(t) = f(x^\star(t) + \tilde{x}(t), u^\star(t) + \tilde{u}(t))$$
$$= f(x^\star(t), u^\star(t)) + \frac{\partial f}{\partial x}(x^\star(t), u^\star(t))\tilde{x}(t) + \frac{\partial f}{\partial u}(x^\star(t), u^\star(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)$$

$\dot{x}^\star$

H.O.T

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(x^\star(t), u^\star(t))\tilde{x}(t) + \frac{\partial f}{\partial u}(x^\star(t), u^\star(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)$$

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Linearization around a trajectory

Hence, for small $(\tilde{x}, \tilde{u})$, approximately

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t)$$

Example: if dim $x = 2$, dim $u = 1$

2×2

$$A(t) = \frac{\partial f}{\partial x}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}(x^\star(t), u^\star(t))$$

$$B(t) = \frac{\partial f}{\partial u}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}(x^\star(t), u^\star(t))$$

2×1

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Linearization around a trajectory

Linearization of the output equation $y(t) = h(x(t), u(t))$ around the nominal output $y^\star(t) = h(x^\star(t), u^\star(t))$:

$\dot{x} = f(x, u)$
$y = h(x, u)$

$$\tilde{y}(t) = C(t)\tilde{x}(t) + D(t)\tilde{u}(t)$$

Second order system i.e. dim y = dim $x = 2$, dim $u = 1$

$$C(t) = \frac{\partial h}{\partial x}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix}(x^\star(t), u^\star(t))$$

$$D(t) = \frac{\partial h}{\partial u}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_2}{\partial u_1} \end{bmatrix}(x^\star(t), u^\star(t))$$

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Time-varying Linear Systems

Example: Time-varying

$$\dot{\tilde{x}} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}\tilde{x}$$

$\lambda_{1,2} = -1$

By just checking the eigenvalues someone would that the origin is stable.

$\dot{\tilde{x}}_2 = -\tilde{x}_2 \Rightarrow \tilde{x}_2(t) = \tilde{x}_2(0)\, e^{-t}$

$\dot{\tilde{x}}_1 = -\tilde{x}_1 + \boxed{e^{2t} e^{-t}}\, \tilde{x}_2(0)$

$= -\tilde{x}_1 + e^{t}\, \tilde{x}_2(0)$

$\tilde{x}_1$ will grow unbounded

However, we can solve first for $\tilde{x}_2$ and then substitute in $\dot{\tilde{x}}_1$ and realize the $\tilde{x}_1$ grows unbounded

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Time-varying Linear Systems

$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$

$\lambda_i[A(t) + A^T(t)] < 0$ $\forall t$ ➡ Stable

$\lambda_i[A(t) + A^T(t)] > 0$ $\forall t$ ➡ Unstable

$\lambda_i[A(t) + A^T(t)] = 0$ $\forall t$ ➡ No conclusion

forall $\forall t$

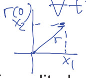
exist $\exists t$

Periodic solutions and Limit Cycles

- A system oscillates when it has a nontrivial periodic solution:

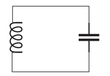
$x(t+T) = x(t), \quad \forall t \geq 0$ for some $T > 0$

Example: $\boxed{\dot{z} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} z}$ polar coordinates $\dot{r} = 0$, $\dot{\theta} = \omega$ $\Rightarrow \theta = \theta_0 + \omega t$; $r(t) = r(0)$ $\forall t > 0$; $r_0 = \sqrt{x_1^2(0) + x_2^2(0)}$



The system has a sustained oscillation of amplitude $r(0)$

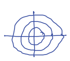
- Harmonic oscillator LC circuit



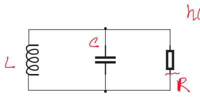
- Small perturbation will destroy the oscillation (e.g. resistance)
- The amplitude depends on the initial conditions

Periodic solutions and Limit Cycles

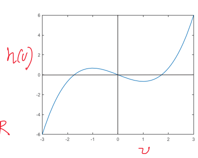
- An **isolated closed** curve in the phase plane
- Closed: periodic solution
- Isolated: limiting nature of the limit cycle, nearby trajectories either converge to or diverge from the limit cycle

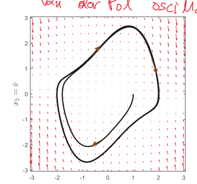


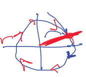
Van der Pol oscillator



$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0$



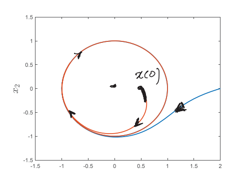


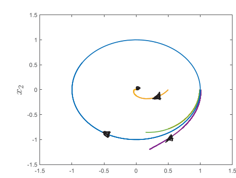


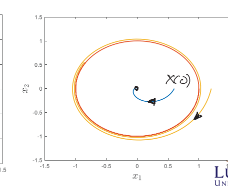
Stability of limit cycles

$t \to \infty$

- **Stable limit cycle:** all trajectories in the vicinity of the limit cycle converge to it
- **Unstable limit cycle:** all trajectories in the vicinity of the limit cycle diverge from it
- **Semi-Stable Limit Cycles:** some of the trajectories in the vicinity converge to it, while the others diverge from it







Stability of limit cycles – matching quiz

check first the stability of the eq. (0,0). If it is unstable the system phase portrait corresponds to Fig 1.

(A) $\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{cases}$ $\sim \dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x$ $(\lambda+1)^2 - 1 = 0 \leadsto \lambda = -1 \pm j$

(B) $\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{cases}$ $\sim \dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x$ $\lambda = -1 \pm j$

(C) $\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{cases}$ $\sim \dot{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x$

only the linearization of the "green" (C) system gives unstable eigenvalues (Re λ > 0) $(\lambda-1)^2 + 1 = 0 \Rightarrow \lambda = 1 \pm j$

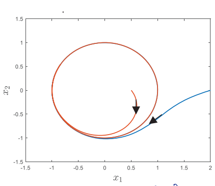


Fig 1. (C)

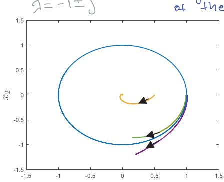


Fig. 2

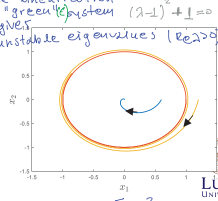


Fig. 3

Stability of limit cycles (linearization around a trajectory)

- Study the stability of the trajectory $(\sin t, \cos t)$

$(*)\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{cases}$

Plug in the trajectory in (*) and verify $\dot{x}^* = f(x^*)$

$x_1 = \sin t$

$x_2 = \cos t$

$\dot{x}_1 = \cos t + 0$ ✓

$\frac{d}{dt}(\sin t) = \cos t$

calculate the Jacobian

$\frac{\partial f}{\partial x}(x_1, x_2) = \begin{bmatrix} (x_1^2+x_2^2-1) + 2x_1^2 & 1 + 2x_1x_2 \\ 2x_1x_2 - 1 & (x_1^2+x_2^2-1) + 2x_2^2 \end{bmatrix}$

$\frac{\partial f}{\partial x}(\sin t, \cos t) = \underbrace{\begin{bmatrix} 2\sin^2 t & 1 + 2\sin t \cos t \\ -1 + 2\sin t \cos t & 2\cos^2 t \end{bmatrix}}_{A}$

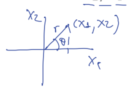
$A + A^T = \begin{bmatrix} 4\sin^2 t & 4\sin t \cos t \\ 4\sin t \cos t & 4\cos^2 t \end{bmatrix}$

$\lambda = 0, \lambda = 1$.

Since $A = \frac{\partial f}{\partial x}$ is time-depedent we cannot conclude by checking the eigenvalues of A

- Change variables $r = \sqrt{x_1^2 + x_2^2}$ $\theta = \arctan x_2/x_1$

$\frac{d}{dt}(x_1^2 + x_2^2) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$



$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{cases}$$

$$\boxed{\dot{r} = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1\dot{x}_1 + x_2\dot{x}_2)}$$

$$\dot{r} = \frac{-1}{\cancel{r}} \cdot r^{\cancel{2}}(r^2-1)^2 = -r(r^2-1)^2$$

$$\dot{\theta} = \frac{1}{x_1^2+x_2^2}(\dot{x}_2 x_1 - x_2\dot{x}_1)$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Intuition

Otherwise calculate $r(t)$ in closed form

if $r(0) > 1$ $\dot{r}(0) < 0$
r is decreasing but will hit $r=1$ and stay there

if $r(0) < 1$ $\dot{r}(0) < 0$
r is decreasing until $r=0$

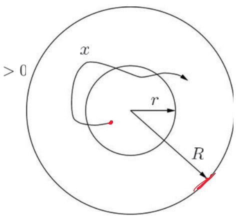
Stability of an equilibrium point

Consider $\dot{x} = f(x)$ where $f(x^*) = 0$

Definition The equilibrium x^* is **stable** if $\boxed{\forall}$ for any $R > 0$, $\boxed{\exists}$ there exists $r > 0$ such that

$$\|x(0) - x^*\| < r \implies \|x(t) - x^*\| < R, \quad \text{for all } t \geq 0$$

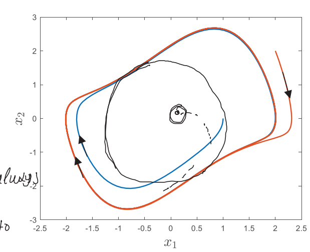
Otherwise the equilibrium point x^* is **unstable**.



- Use the term “stable (unstable) system” only for linear systems
- A nonlinear system have more than one equilibrium points that each one can be either stable or unstable
- Unstable equilibrium does not mean unbounded trajectories

Unstable equilibrium does not mean unbounded trajectories

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1$$



For R defining cycles in the limit cycle the trajectory will always escape R even if it starts very close to equilibrium.

Local vs Global Stability of Equilibrium

Definition The equilibrium x^* is **locally asymptotically stable (LAS)** if it

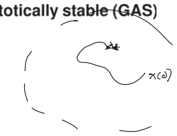
1) is stable

2) there exists $r > 0$ so that if $\|x(0) - x^*\| < r$ then

$$x(t) \longrightarrow x^* \quad \text{as} \quad t \longrightarrow \infty.$$

Definition The equilibrium is said to be **globally asymptotically ~~stable~~ (GAS)** if it is LAS and for all $x(0)$ one has

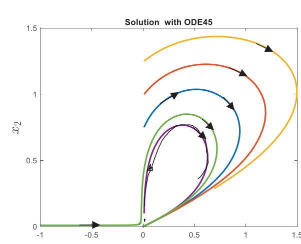
$$x(t) \to x^* \text{ as } t \to \infty.$$



Convergent but not stable

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}$$

$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}$$



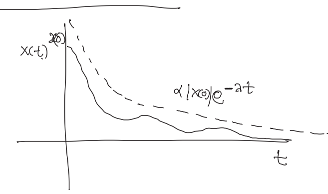
Exponential stability

Definition The equilibrium is said to be **exponentially stable (ES)** if there exists to positive constants a and λ such that:

$$\|x(t) - x^*\| \leq a\|x(0) - x^*\|e^{-\lambda t}, \quad \text{for all } t \geq 0$$

Global: For all $x(0)$

Local: For $\|x(0) - x^*\| < r$



FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 4: Describing function analysis

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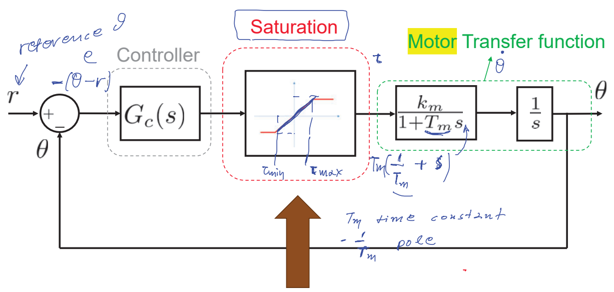
Outline

- How to obtain a describing function for a nonlinear element in an "almost" linear system
- Prediction of oscillations based on extended Nyquist Criterion and the describing function of the nonlinearity

Motivation: Nonlinearities in the control system



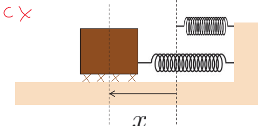
Nonlinear element

Motivation: Nonlinearities

- The physical system (the plant) may contain nonlinearities

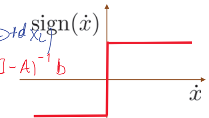
$m\ddot{x} + d\dot{x} + kx + k_h x^3 = 0 \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{m}(kx_1 + k_h x_1^3 + d x_2) \end{cases}$

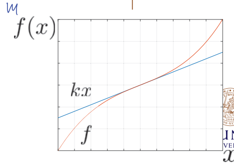
$y = cx$ $\quad H(s) = c(sI-A)^{-1}b$



$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{m}(kx_1 + dx_2) + \frac{1}{m}u \end{cases}$ $\quad u = k_h x_1^3$

L7: $ms^2X(s) + dsX(s) + kX(s) = -U(s)$



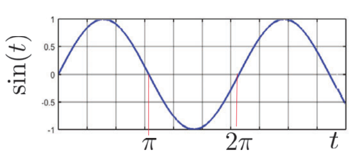


Coulomb Friction $\quad -k_c\text{sign}(\dot{x})$

Hardening spring $\quad f = kx + k_h x^3$

Motivation: prediction of persistent oscillations (limit cycles)

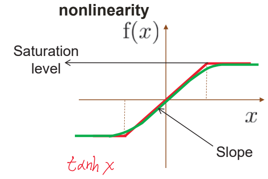
- Oscillations can be desirable: electronic oscillators used in laboratories.



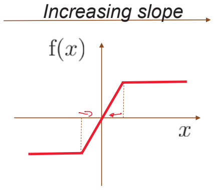
- Oscillations are undesirable
  - Oscillations are a sign of instability, tend to cause poor control accuracy
  - Constant oscillations can increase wear or even cause mechanical failure

Nonlinearities: Single-valued nonlinearities

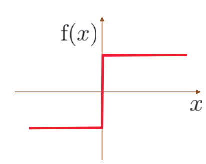
- **Saturation nonlinearity**



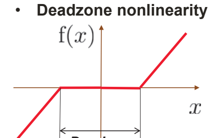
Increasing slope



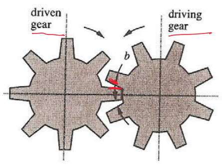
- **On-Off (relay) nonlinearit**

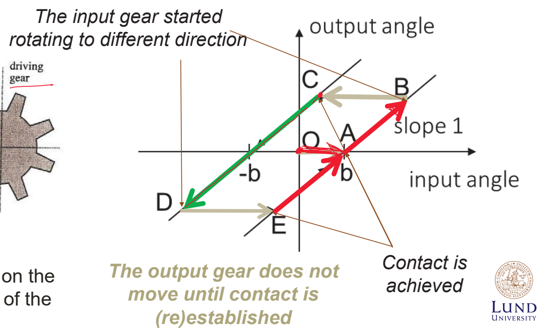


- **Deadzone nonlinearity**



Nonlinearities: Backlash



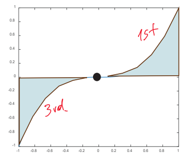


- Multi-valued
- The output depends on the input and the history of the input

Odd and even functions

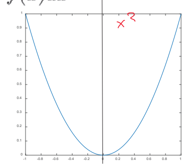
- Odd function $f(-x) = -f(x)$

$\int_{-a}^{a} f(x)dx = 0$



- Even function $f(-x) = f(x)$

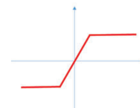
$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$



- *Examples*

Deadzone

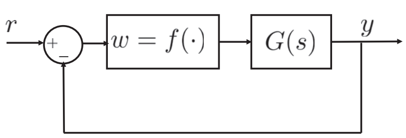
Saturation

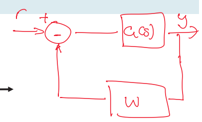


Odd functions with

$xf(x) \geq 0$

Describing function analysis

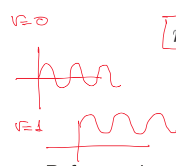


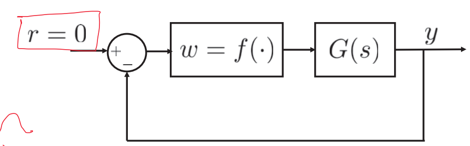


- Assumptions
  - Single, odd, time-invariant nonlinear element $f(\cdot)$
  - Low-pass transfer function $G(s)$
- Replace the nonlinearity with a quasi-linear component
- Use tools from linear control systems design to examine the existence of oscillations

Describing function analysis

- Form of the nonlinear system

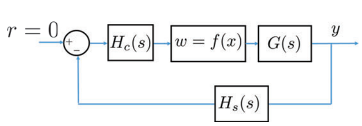




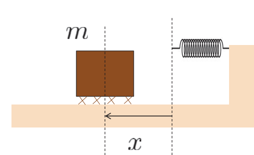
- Reference is set zero to study self-sustained oscillations
- "almost" linear system or genuinely nonlinear system (written as shown in the block diagram)

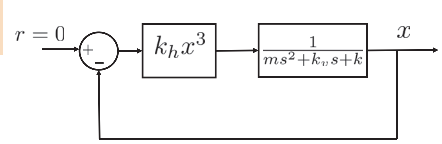
"Almost" linear systems

- Linear Control Design and linear system
- Implementation involves hard nonlinearities, e.g. actuator saturation or sensor dead-zones
- Contain hard nonlinearities in the control loop but are otherwise linear



Quiz: Write the nonlinear system in a feedback form where the nonlinearity is in a block



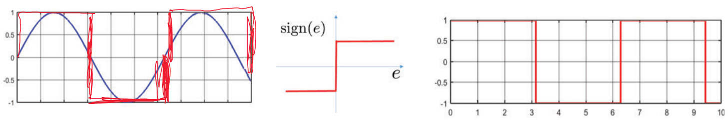


Viscous Friction $-k_v\dot{x}$

Hardening spring $f = -kx - k_h x^3$

Fourier Transformation

Input $e(t) = A\sin(\omega t)$ **Output** $w(t) = f(e) = f(A\sin(\omega t))$



Output – Periodic function $w(t+T) = w(t)$

Fourier Transformation $w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)d(\omega t)$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(n\omega t)d(\omega t)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(n\omega t)d(\omega t)$

0 for odd w

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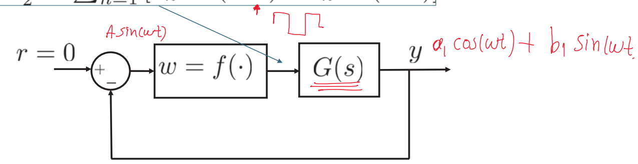
The linear transfer function as a low-pass filter

$w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]$



- If the transfer function is acting as a low pass filter the output y will be mainly affected by the first harmonic of w

$$w(t) = w_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t)$$

- The method is based on approximations (heuristic)

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The linear transfer function as a low-pass filter

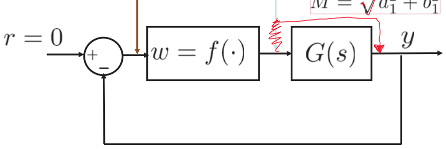
- *"Filtering" Assumption: the first harmonic is taken as output of the nonlinear block*

$A\sin(\omega t)$

$w_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t)$

$w_1(t) = M\sin(\omega t + \phi)$

$M = \sqrt{a_1^2 + b_1^2}$ $\phi = \arctan(a_1, b_1)$



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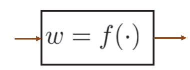
Describing Function

Input of the nonlinear element

$Ae^{j\omega t}$



Output of the nonlinear element

$w_1(t) = Me^{(j\omega t+\phi)}$

$w_1(t) = (b_1 + ja_1)e^{j\omega t}$

- Describing function definition

$N(A,\omega) = \frac{\text{Output}}{\text{Input}}$ ➡ $N(A,\omega) = \frac{Me^{(j\omega t + j\phi)}}{Ae^{j\omega t}} = \frac{M e^{j\omega t} e^{j\phi}}{A e^{j\omega t}}$

➡ $N(A,\omega) = \boxed{\frac{M}{A}e^{j\phi}}$

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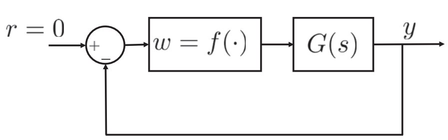
Describing Function (cont.)

$N(A,\omega) = \frac{M}{A}e^{j\phi}$

$N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$



- Extension of the notion of frequency response for systems with nonlinearities
- Depends on the amplitude of the input signal in contrast to the frequency response for linear systems

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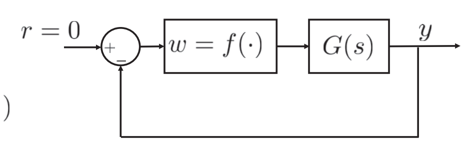
Describing Function –special cases

$N(A,\omega) = \frac{M}{A}e^{j\phi}$

$N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$



- It is **real** and **independent of the frequency** when the non-linearity is **single-valued**
- Why? $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(\omega t)d(\omega t)$ ⟶ • Imaginary part

  $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(\omega t)d(\omega t)$ ⟶ • Real part

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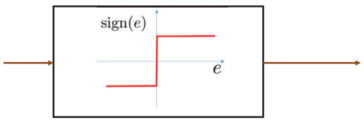
Describing Function – Example

$N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$

$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(\omega t)d(\omega t)$

$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(\omega t)d(\omega t)$



Describing Function – Example

$w(t) = \text{sign}\,(\sin\omega t)$

$N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$

$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(\omega t)d(\omega t)$

$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(\omega t)d(\omega t)$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}[\sin(\omega t)]\cos(\omega t)d(\omega t) = 0$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\sigma d\sigma = -\frac{1}{\pi}\int_{-\pi}^{0} \cos\sigma d\sigma + \frac{1}{\pi}\int_{0}^{\pi} \cos\sigma d\sigma = 0$$

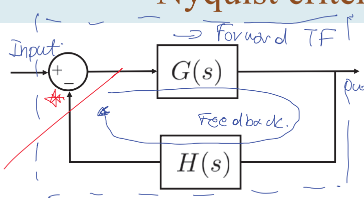
$N(A,\omega) = \frac{b_1}{A} = \frac{4}{A\pi}$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}[\sin(\omega t)]\sin(\omega t)d(\omega t)$$

$$b_1 = \frac{2}{\pi}\int_{0}^{\pi} \sin\sigma d\sigma = \frac{2}{\pi}[-\cos\sigma]_0^{\pi} = \frac{4}{\pi}$$

$$\boxed{N(A,\omega) = \frac{4}{A\pi}}$$

Nyquist criterion: Definitions



$1 + \frac{s+1}{(s-1)(s-2)} = \frac{(s-1)(s-2) + (s+1)}{(s-1)(s-2)}$

- The **characteristic equation** of the system:

$$\Delta(s) = 1 + G(s)H(s) = 0$$

- Poles of $\Delta(s)$→ poles the OLS system
- Zeros of $\Delta(s)$ → poles of the CLS system

- Closed loop Transfer Function

$$\frac{G(s)}{1+G(s)H(s)}$$

- Open loop Transfer Function

$$G(s)H(s) = \frac{a_m s^m + \cdots + a_1 s + a_0}{b_n s^n + \cdots + b_1 s + b_0}$$

$m \le n$ for proper/strictly proper transfer function

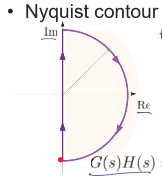
- Example:

$$G(s)H(s) = \frac{s+1}{(s-1)(s-2)}$$

Nyquist contour and plot

- Nyquist contour

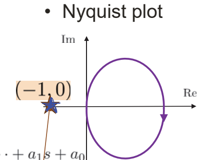
$H(s)\,G(s)$



$G(s)H(s) = \frac{a_m s^m + \cdots + a_1 s + a_0}{b_n s^n + \cdots + b_1 s + b_0}$

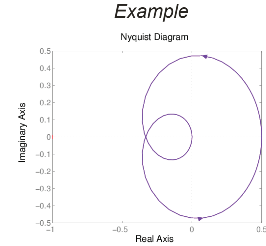
A path that encircles the right-half s plane

- Nyquist plot



Number of clockwise encirclements of the point (-1,0)

Example



$$G(s)H(s) = \frac{s+1}{(s-1)(s-2)}$$

$G(j\omega)H(j\omega) = R(\omega) + jI(\omega)$

- Nyquist Criterion

$$P_{CL} - P_{OL} = \text{N(-1,0)}$$

Nyquist Criterion

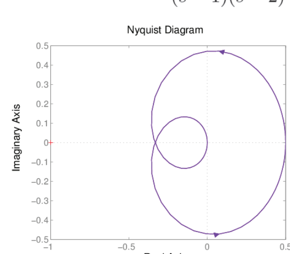
- *The number of unstable Closed Loop Poles is equal to the number of open loop poles with positive real part plus the number of clockwise encirclements of the point (-1,0)*

$$P_{CL} = \text{N}(-1,0) + P_{OL}$$

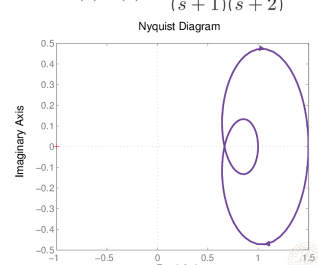
- *Given a stable open loop system, the closed loop is stable if the Nyquist plot of the open loop system does not encircle the point (-1,0).*

Nyquist Criterion: Quiz

$$G(s)H(s) = \frac{s+1}{(s-1)(s-2)}$$

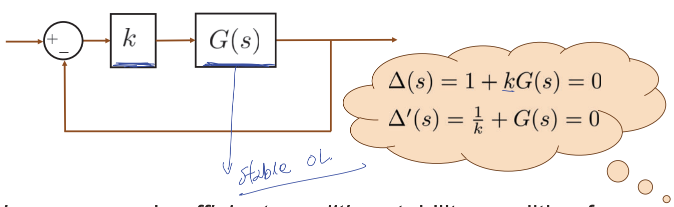


$$G(s)H(s) = \frac{(s+1)^2 + 2}{(s+1)(s+2)}$$



Stable or Unstable?

Nyquist Criterion

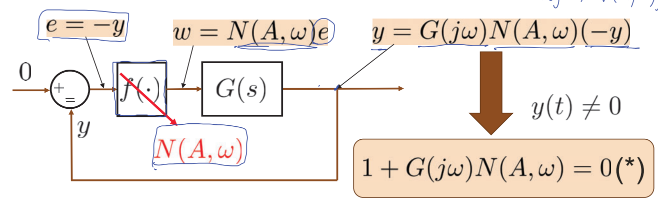


$$\Delta(s) = 1 + kG(s) = 0$$
$$\Delta'(s) = \tfrac{1}{k} + G(s) = 0$$

Necessary and *sufficient condition* stability condition for systems for stable open-loop systems:
The Nyquist plot does not encircle the point ***-1/k***

Extension of Nyquist Criterion for Describing Function Analysis (Existence of oscillations)

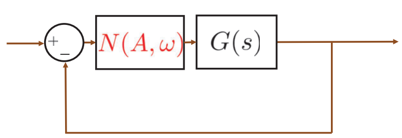
- Assume that there exists self-sustained oscillations $(1+G(j\omega)N(A,\omega))y=0$



$$1+G(j\omega)N(A,\omega)=0\,(*)$$

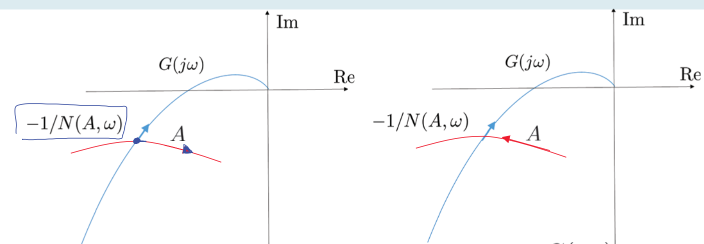
- The amplitude and frequency must satisfy (*) – Harmonic balance
- If (*) has no solutions then there are no oscillations in the system

Extension of Nyquist Criterion for Describing Function Analysis (Stability of oscillations)



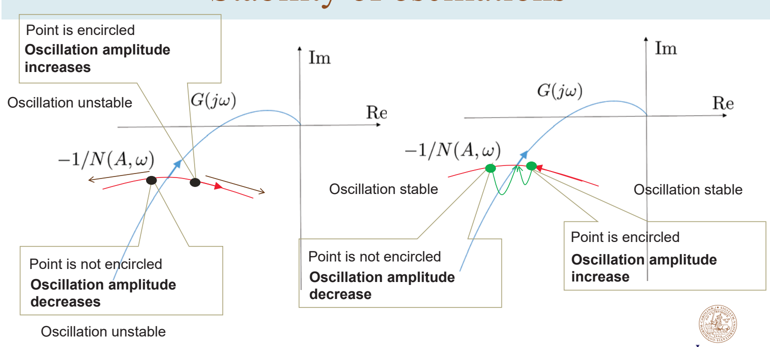
Necessary and *sufficient condition* stability condition for systems ~~for systems~~ with stable (open-loop) linear part:
The Nyquist plot does not encircle the point $\frac{-1}{N(A,\omega)}$

Stability of oscillations



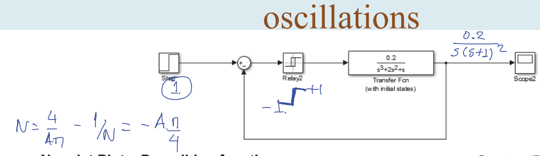
- Permanent oscillation: Common point of the Nyquist diagram $G(j\omega)$ and the plot $\frac{-1}{N(A,\omega)}$
- Stability of the oscillation: Does the oscillation continue after a small perturbation in A?

Stability of oscillations

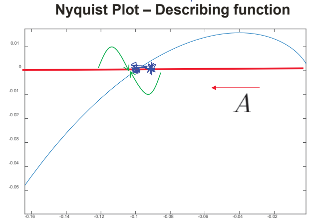


Stability of the oscillation: Does the oscillation continue after a small perturbation in A?

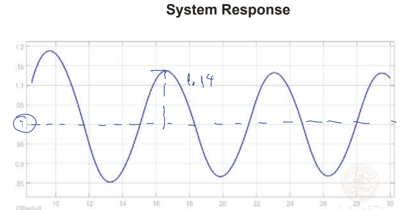
Example – Prediction and stability of persistent oscillations



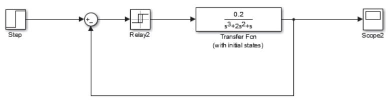
Nyquist Plot – Describing function



System Response



Example – Prediction of oscillations



$$G(s)=\frac{K}{s(1+s)^2}$$

$$K=0.2$$

$$G(j\omega)=\frac{K}{j\omega(1+j\omega)^2}=\frac{K}{-2\omega^2+j\omega(1-\omega^2)}\,\frac{-2\omega^2-j\omega(1+\omega^2)}{-2\omega^2-j\omega(1+\omega^2)}$$

$s=j\omega$

$$=\boxed{-\frac{2K\omega^2}{4\omega^4+\omega^2(1-\omega^2)^2}}-j\boxed{\frac{K\omega(1-\omega^2)}{4\omega^4+\omega^2(1-\omega^2)^2}}$$

Real for $\boxed{\omega=1\mathrm{rad/s}}$

$$\left.\begin{aligned}G(j1)&=-\frac{K}{2}\\ N(A)&=\frac{4}{A\pi}\end{aligned}\right\}\quad \boxed{G(j\omega)=-\frac{1}{N(A)}}\quad A=\frac{2K}{\pi}$$

For $K=0.2$ $\boxed{A=0.127}$

Describing function analysis: pitfalls

- DF analysis may predict a limit cycle, even if it does not exist.
- A limit cycle may exist, even if DF analysis does not predict it.
- The predicted amplitude and frequency are only approximations and can be far from the true values.

FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 5: Lyapunov stability I

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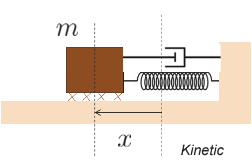
Outline

- Physics based motivation
- Lyapunov function candidates
- Local Lyapunov Stability
- Global Lyapunov Stability
- Lyapunov stability for linear systems
- Lyapunov stability with linearization

Example (nonlinear spring with external force)



- Differential Equation

$$m\ddot{x}+\underbrace{b|\dot{x}|\dot{x}}_{\text{Nonlinear damping}}+\underbrace{k_0x+k_1x^3}_{\text{Hardening spring}}=0$$

- State space representation

Position: $x_1=x$ **Velocity:** $x_2=\dot{x}$

- *Plug in the system dynamics*
- *Derivative along the system trajectories*

$$\boxed{m\ddot{x}=-b|\dot{x}|\dot{x}-k_0x-k_1x^3}$$

- Energy $E(x,\dot{x})=\underbrace{\frac{m\dot{x}^2}{2}}_{\text{Kinetic}}+\underbrace{\int_0^x F_{spring}\,ds}_{\text{Potential}}$

$$\int_0^x(k_0s+k_1s^3)\,ds=\frac{1}{2}k_0x^2+\frac{1}{4}k_1x^4$$

$$\frac{dE}{dt}=\frac{\cancel{2}m\dot{x}\ddot{x}}{\cancel{2}}+F_{spring}\dot{x}\quad \frac{d}{dt}E(x,\dot{x})=\boxed{m\ddot{x}}\dot{x}+k_0x\dot{x}+k_1x^3\dot{x}$$

$$\frac{d}{dt}E(x,\dot{x})=-b|\dot{x}|\dot{x}^2\leq 0$$

Lyapunov function candidates

$V(x):\Omega\to\Re$ is a $\boxed{C^1\text{ function}}$, $\frac{\partial V}{\partial x}$ exists + continuous

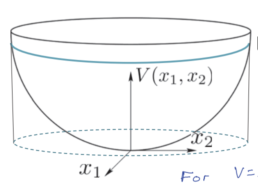
$V(x_1,x_2,\dots)$

$V(x)$ is positive definite if $V(x)>0\ \forall x\neq 0$, and $V(x)=0$ only if $x=0$.

$V(x)$ is positive semi-definite if $V(x)\geq 0\ \forall x\in\Omega$.

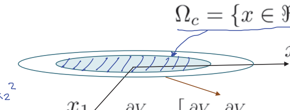
example $V(x)=x_1^2+x_2^2$

Level sets $x_1^2+x_2^2=c$



Lyapunov level set where $V=c$ (c constant)

$$\Omega_c=\{x\in\Re^n:\ V(x)\leq c\}$$



For $V=x_1^2+x_2^2$

$\frac{\partial V}{\partial x}=[2x_1\ \ 2x_2]$

$$\frac{\partial V}{\partial x}=\left[\frac{\partial V}{\partial x_1}\ \frac{\partial V}{\partial x_2}\ \dots\ \frac{\partial V}{\partial x_n}\right]$$

Lyapunov function candidates – positive definite functions

$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $x^*=\begin{bmatrix}0\\1\end{bmatrix}$, $x^{**}=\begin{bmatrix}0\\1\end{bmatrix}$ $V(x^*)=V(x^{**})=0$

$V(x)=x_1^2$ → positive semidefinite

$V(x)=x_1^2+ax_2^2$ if $a>0$ p.d

$V(x)=x_1^2+(x_1-x_2)^2$ $V(x)>0$ $V(x)=0$ when $\begin{cases}x_1=0\\x_2=x_1\end{cases}$, when $x=0$

$V(x)=(x_1+x_2)^2$ → positive semidefinite $V(x)=0$ when $x_1=-x_2$

$V(x)=\int_0^{x_1}h(y)dy+x_2^2$
$V(x)=h^2(x_1)+x_2^2$

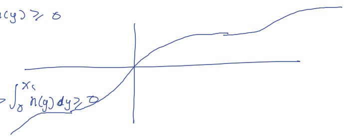
$h(y)$ *increasing function with* $h(0)=0$

$h(y)=0$ unique solution $y^*=0$

$y\,h(y)\geq 0$

$V=\int_0^{x_1}h(y)dy+\underline{x_2^2}$
↳ zero for $x_2=0$

$y>0 \Rightarrow h(y)\geq h(0) \Rightarrow h(y)\geq 0 \Rightarrow \int_0^{x_1}h(y)dy\geq 0$
increasing $h(0)=0$



Lyapunov function candidates – quadratic forms

For a symmetric matrix $M=M^T$ all eigenvalues are real and:

- $x^TMx>0,\ \forall x\neq 0 \iff M$ positive definite $\iff \lambda_i(M)>0,\ \forall i$
- $x^TMx\geq 0,\ \forall x \iff M$ positive semi-definite $\iff \lambda_i(M)\geq 0,\ \forall i$

- For a symmetric matrix $M=M^T$

$\|x\|^2=x_1^2+\dots+x_n^2$

$$\lambda_{\min}(M)\|x\|^2 \leq x^TMx \leq \lambda_{\max}(M)\|x\|^2,\quad \forall x$$

Proof idea: factorize $M=U\Lambda U^T$, unitary U (i.e., $\|Ux\|=\|x\|\ \forall x$), $\Lambda=\text{diag}(\lambda_1,\dots,\lambda_n)$

- For any matrix M

$$\|Mx\| \leq \sqrt{\lambda_{\max}(M^TM)}\|x\|,\quad \forall x$$

$V=x_1^2=\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{M}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

eigs$(M)=(0,1)$

$M\rightarrow$ positive semidefinite

$V=x_1^2+(x_1-x_2)^2=\begin{bmatrix}x_1\\x_2\end{bmatrix}^T\underbrace{\begin{bmatrix}2&-1\\-1&1\end{bmatrix}}_{M}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$\det(M)=1>0$
$2,1>0$ ↦ eigs >0

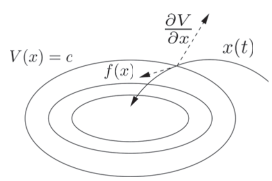
$M\Rightarrow$ positive def.

Differentiating Lyapunon function candidates along trajectories

$$\dot V(x)=\frac{\partial V}{\partial x}\dot x=\sum_i\frac{\partial V}{\partial x_i}f_i(x)$$

$\dot x=f(x)$

$$\frac{\partial V}{\partial x}=\left[\frac{\partial V}{\partial x_1},\frac{\partial V}{\partial x_2},\dots\frac{\partial V}{\partial x_n}\right]$$



Example: $V=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$

$$\begin{cases}\dot x_1= & -x_1+x_1x_2\\ \dot x_2= & -x_2-x_1^2\end{cases}$$

$\dot V=x_1\dot x_1+x_2\dot x_2$
$=-x_1^2+\cancel{x_1^2x_2}-x_2^2-\cancel{x_1^2x_2}$
$=-x_1^2-x_2^2$

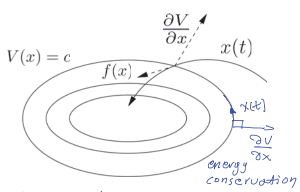
$$\begin{cases}\dot x_1= & -x_2+x_1x_2\\ \dot x_2= & x_1-x_1^2\end{cases}$$

$\dot V=[x_1\ \ x_2]\begin{bmatrix}-x_2+x_1x_2\\x_1-x_1^2\end{bmatrix}$
$=\cancel{-x_1x_2}+\cancel{x_1^2x_2}+\cancel{x_2x_1}-\cancel{x_1^2x_2}=0$

Energy conservation and dissipation

$$\dot V(x)=\frac{\partial V}{\partial x}\dot x=\sum_i\frac{\partial V}{\partial x_i}f_i(x)$$

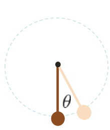
$$\frac{\partial V}{\partial x}=\left[\frac{\partial V}{\partial x_1},\frac{\partial V}{\partial x_2},\dots\frac{\partial V}{\partial x_n}\right]$$



$a^Tb=a\cdot b=\|a\|\|b\|\cos\theta$ angle between
↳ a,b

- Energy conservation: $\dot V(x)=\frac{\partial V}{\partial x}f(x)=0$, i.e., $f(x)\perp\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial x}$: normal to the level surface $V(x)=c$
  ↳ inner product
- Energy dissipation: $\dot V(x)=\frac{\partial V}{\partial x}f(x)\leq 0$, i.e., $f(x)$ $\frac{\partial V}{\partial x}$ forms an angle bigger than $\pi/2$ smaller than π
  (equal)

Energy conservation and dissipation (pendulum)



$$m\ell^2\ddot\theta+k\ell\dot\theta+mg\ell\sin\theta=0$$

$$\begin{cases}\dot x_1= & x_2\\ \dot x_2= & -\frac{b}{m}x_2-\frac{g}{\ell}\sin x_1,\quad b=\frac{k}{\ell}\end{cases}$$

$$\underbrace{V(x_1,x_2)=E(\theta,\dot\theta)}_{\text{Energy}}=\underbrace{\frac{1}{2}m\ell^2\dot\theta^2}_{\text{kinetic}}+\underbrace{mg\ell(1-\cos\theta)}_{\text{potential}}$$

No friction $b=0$, lossless mechanical system

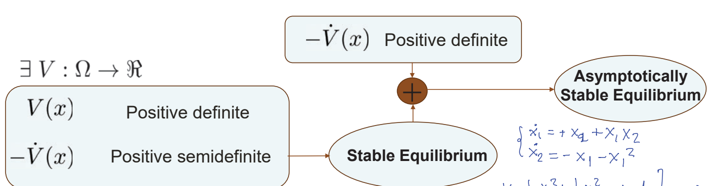
Friction $b>0$, damping

$\dot E(\theta,\dot\theta)=m\ell^2\dot\theta\ddot\theta+mg\ell\sin\theta\,\dot\theta=\dots=-b\ell^2\dot\theta^2$

if $b=0$ $\dot E=0$

Local stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$\dot x=f(x),\ f(x^\star)=0\qquad x^\star=0\in\Omega\subset\Re^n$



$\begin{cases}\dot x_1=+x_2+x_1x_2\\ \dot x_2=-x_1-x_1^2\end{cases}$

$V=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$ p.d
$\dot V=0$ } (0,0) stable

$\begin{cases}\dot x_1=-x_1+x_1x_2\\ \dot x_2=-x_2-x_1^2\end{cases}$, $V=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$

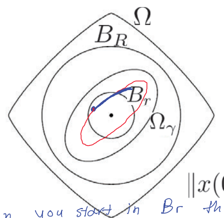
eq. point
$x_1=x_2=0$

Stability of $(0,0)$
V positive definite
$\dot V=-x_1^2-x_2^2<0$ $-\dot V$ pos. def. } ⇒ (0,0) asympt. stable

Sketchy proof of the basic Lyapunov Theorem on stability

$x^\star = 0 \qquad \forall R,\ \exists\, r(R),\ x(t)$ starts in $B_r = \{x \in \Re : \|x\| \le r\}$

remains in $B_R = \{x \in \Re : \|x\| \le R\} \subset \Omega$



Choose $\Omega_\gamma = \{x \in \Re : V(x) \le \gamma\} \subset B_R$

Choose $B_r = \{x \in \Re : \|x\| \le r\} \subset \Omega_\gamma$

$\dot V(x(t)) \le 0$

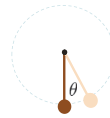
$\|x(0)\| < r$ ➡ $V(x(t)) \le V(x(0)) := a,\ \forall t \ge 0$

$x(t) \in \Omega_\gamma\ ,\ \forall t \ge 0$

$x(t) \in B_R,\ \forall t \ge 0$

When you start in B_r the "worst" that can hapen is to stay in $V(x) \le V(x(0)) = \alpha < \gamma$ the trajectory is in $\Omega_\gamma \subset B_R$

Stability analysis of eq. for the pendulum



$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

Equilibrium $x_2 = 0$, $\sin x_1 = 0 \Rightarrow x_1 = \pm n\pi$

By assuming $\Omega = \{|x_1| < \pi,\ x_2 < M\}$ $(x_1, x_2) = (0,0)$ is the only eq. in Ω

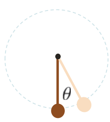
$$V(x_1, x_2) = \frac{1}{2}m\ell^2 x_2^2 + mg\ell(1 - \cos x_1)$$ p.d in Ω

for $|x_1| < \pi$ $(1-\cos x_1) > 0$

$$\dot V(x_1, x_2) = [mg\ell \sin x_1 \ \ m\ell^2 x_2]^T \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} = \dots = -b\ell^2 x_2^2 \le 0$$

Based on the Theorem, the conclusion we make is that the eq. $(0,0)$ is stable

Stability analysis of eq. for the pendulum



$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

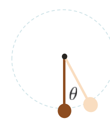
$$V(x_1, x_2) = \frac{1}{2}m\dot x_2^2 + \frac{mg}{\ell}(1 - \cos x_1)$$

$$\dot V(x_1, x_2) = [\frac{mg}{\ell}\sin x_1 \ \ m x_2]^T \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} = \dots = -b x_2^2$$

* We cannot make any conclusion for asymptotic stability (stability and convergence)

* We need to further investigate by e.g. considering another Lyapunov function candidate.

Stability analysis of eq. for the pendulum



$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

$$V(x_1, x_2) = \frac{1}{2}x^T P x + mg\ell(1 - \cos x_1), \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$V = \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + mg\ell(1 - \cos x_1)$$

p.d. for $p_{11}p_{22} - p_{12}^2 > 0 \quad p_{11} > 0$

$$\dot V = p_{11}x_1\dot x_1 + p_{12}x_1\dot x_2 + p_{21}x_2\dot x_1 + p_{22}x_2\dot x_2 + mg\ell \sin x_1 x_2$$

Differentiate V

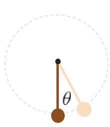
Substitute the system dynamics

$$\dot V = p_{11}x_1x_2 + p_{12}x_1\left(-\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1\right) + p_{21}x_2^2 + p_{22}x_2\left(-\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1\right) + mg\ell \sin x_1 x_2$$

$$\dot V = (p_{11} - p_{12}\frac{b}{m})x_1x_2 - p_{12}x_1\frac{g}{\ell}\sin x_1 + (p_{21} - p_{22}\frac{b}{m})x_2^2 + (mg\ell - p_{22}\frac{g}{\ell})\underline{x_2 \sin x_1}$$

group the terms that involve the states

Stability analysis of eq. for the pendulum



$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

The terms underlined have known sign in $\Omega = \{|x_1| < \pi, |x_2| < M\}$

$$V(x_1, x_2) = \frac{1}{2}x^T P x + mg\ell(1 - \cos x_1), \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$V = \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + mg\ell(1 - \cos x_1)$$

The terms in boxes are sign indefinite (should be cancelled)

We don't know their sign if we don't know the functions

$p_{11}p_{22} - p_{12}^2 > 0 \qquad 0 < p_{12} < b\ell^2$

$$\dot V = (p_{11} - p_{12}\frac{b}{m})\boxed{x_1x_2} - p_{12}x_1\frac{g}{\ell}\sin x_1 + (p_{21} - p_{22}\frac{b}{m})x_2^2 + (mg\ell - p_{22}\frac{g}{\ell})\boxed{x_2 \sin x_1}$$

$\boxed{p_{11} = p_{12}\frac{b}{m}}$

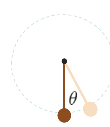
$p_{12} > 0$ $x_1 \sin x_1 > 0$ in Ω $\quad p_{12} < p_{22}\frac{b}{m}$

$0 < p_{12} < b\ell^2$

$\boxed{p_{22} = m\ell^2}$

$p_{11} = \frac{b^2\ell^2}{2m} \qquad p_{12} = \frac{b\ell^2}{2} \qquad > 0 \qquad p_{22} = m\ell^2$

Stability analysis of eq. for the pendulum



$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

$$V(x_1, x_2) = \frac{1}{2}x^T P x + mg\ell(1 - \cos x_1), \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$V = \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + mg\ell(1 - \cos x_1)$$

$p_{11} = \frac{b^2\ell^2}{2m} \qquad p_{12} = \frac{b\ell^2}{2}$

$p_{22} = m\ell^2$

$$\dot V = -\frac{g\ell g}{2}x_1 \sin x_1 - \frac{b\ell^2}{2}x_2^2$$

$\left.\begin{matrix} V \text{ p d} \\ -\dot V \text{ p d} \end{matrix}\right\} \Rightarrow (0,0)$ asymptotically stable equilibrium

Global stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$

$\exists\, V : \Re^n \to \Re$

$V(x)$ Positive definite

$-\dot{V}(x)$ Positive semidefinite → Stable Equilibrium

$-\dot{V}(x)$ Positive definite ⊕ Stable Equilibrium → Asymptotically Stable Equilibrium

Is it enough to consider $\Omega = \Re^n$?

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Study the stability of the eq. point

Example:

$$\begin{cases} \dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{cases}$$

$$V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2 \qquad \text{p.d}$$

$$\dot{V} = \frac{x_1\dot{x}_1}{(1+x_1^2)^2} + x_2\dot{x}_2 = -\frac{6x_1^2}{(1+x_1^2)^4} - \frac{2x_2^2}{(1+x_1^2)^2} \qquad -\dot{V}\ \text{pd} \Bigg\} \to$$

asymptotically stable global ?

→ check the next slide

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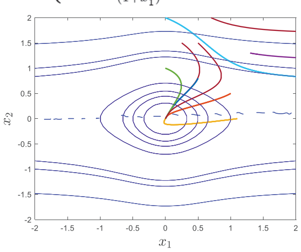
Radially unbounded functions

Example:

$$\begin{cases} \dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{cases}$$



Radially unbounded function: $V(x) \to \infty$ as $\|x\| \to \infty$

Is $V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$ radially unbounded? $V = \frac{1}{2}$ * is unbounded

No

If $\|x\| \to \infty$ e.g. $x_1 \to \infty$, $x_2 \to K$

then $V \to \frac{1}{2} + \frac{1}{2}k^2$

* closed Lyapu sets $V < \frac{1}{2}$

Find common points of $V = c$ with $x_2 = 0$ $\Rightarrow \frac{1}{2}\frac{x_1^2}{1+x_1^2} = c$

$x_1^2 - 2c x_1^2 = 2c$

$x_1^2 = \frac{2c}{1-2c}$

when $c = \frac{1}{2}$ no common points

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Lyapunov function candidates for global stability – radially unbounded functions

$V(x) = x_1^2 + a x_2^2$

$V(x) = x_1^2 + (x_1 - x_2)^2$

$V(x) = \int_0^{x_1} h(y)dy + x_2^2$

$V(x) = h^2(x_1) + x_2^2$

$\Big\}$ $h(y)$ *increasing function with* $h(0) = 0$

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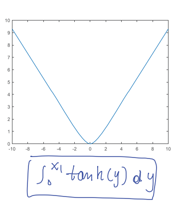
Lyapunov function candidates – radially unbounded functions

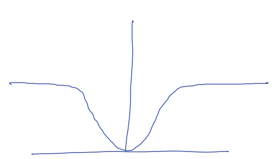
$V(x) = \int_0^{x_1} h(y)dy + x_2^2$

$V(x) = h^2(x_1) + x_2^2$

$\Big\}$ $h(y)$ *increasing function with* $h(0) = 0$



$\boxed{\int_0^{x_1} \tanh(y)\,dy}$



$(\tanh y)^2$

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Global stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$

$\exists\, V : \Re^n \to \Re$

$V(x)$ Positive definite

$V(x)$ Radially unbounded

$-\dot{V}(x)$ Positive semidefinite → Global Stable Equilibrium

$-\dot{V}(x)$ Positive definite ⊕ Global Stable Equilibrium → Global Asymptotically Stable Equilibrium

$\Omega = \Re^n$

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Stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$ $\qquad \Omega(\Re^n)$

$\exists\, V : \Omega(\Re^n) \to \Re$

$V(x)$	Positive definite
$V(x)$	Radially unbounded
$-\dot{V}(x)$	Positive semidefinite

$\Omega(\Re^n)$

→ **Stable Equilibrium**

$-\dot{V}(x)$ Positive definite

+ → **Asymptotically Stable Equilibrium**

Lyapunov stability analysis - comments

- The conditions of the Theorem are only sufficient

If conditions are not satisfied:

It does not mean that the equilibrium is unstable.

It means that the chosen Lyapunov function does not allow to make a conclusion

It requires further investigation

- ✓ *try to find another Lyapunov function*
- ✓ *Use other Theorems ☺*

Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

$$\dot{x} = Ax \; = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

Eigenvalues of $A : \{-1, -3\}$

$\Rightarrow$ (global) asymptotic stability.

Try to prove stability with:

$V(x) = \|x\|^2 = x^T x = x_1^2 + x_2^2$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= -x_1^2 + 4x_1x_2 - 3x_2^2$$
$$= -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det(M) = 3 - 4 = -1$

$-\dot{V}$ is not p.d.

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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 6: Lyapunov stability II

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Stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$ $\qquad \Omega(\Re^n)$

$\exists\, V : \Omega(\Re^n) \to \Re$

$V(x)$	Positive definite	$V(x^\star) = 0$, $V(x) > 0 \ \forall x \neq x^\star$
$\boxed{V(x)}$	$\boxed{\text{Radially unbounded}}$	
$-\dot{V}(x)$	Positive semidefinite	$-\dot{V}(x) \geq 0$

$\hookrightarrow$ It defines a set of equilibria

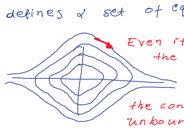
$\Omega(\Re^n)$

→ **Stable Equilibrium**

$-\dot{V}(x)$ Positive definite

+ → **Asymptotically Stable Equilibrium**

$V(x) \to \infty$ as $\|x\| \to \infty$



Even if $\dot{V}(x) = 0$ the trajectory can escape to infinity if the condition on radially unboundedness is not satisfied

Lyapunov stability analysis - comments

- The conditions of the Theorem are only sufficient

If conditions are not satisfied:

It does not mean that the equilibrium is unstable.

It means that the chosen Lyapunov function does not allow to make a conclusion

It requires further investigation

- ✓ *try to find another Lyapunov function*
- ✓ *Use other Theorems ☺*

Outline

- Softer conditions
- Convergence rate (exponential stability)
- Invariant Sets
- Region of attraction
- Asymptotic stability of invariant sets
- Lyapunov stability for linear systems

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Asymptotic Stability (softer condition on $\dot{V}$)

Barbashin, Krasovskii Theorem (LaSalle Invariance Principle is more general and proved afterwards, we can call this LaSalle Theorem)

Theorem: Let $\dot{x} = f(x)$ and $f(x^\star) = 0$. If there exists a $\mathcal{C}^1$ function $V : \Re^n \to \Re$ such that

we call this LaSalle Theorem and we use it when

(1) $V(x) > 0$ for all $x \neq x^\star$ and $V(x^\star) = 0$

(2) $V(x) \to \infty$ as $\|x\| \to \infty$

(3) $\boxed{\dot{V}(x) \leq 0 \text{ for all } x}$

(4) No solution of $\dot{x} = f(x)$ can stay identically in $E = \left\{x \in \Re^n : \dot{V} = 0\right\}$ except of $x = x^\star$

1. Find the solution corresponding to $\dot{V} \equiv 0$
2. Substitute in $\dot{x} = f(x)$ and show that corresponding to $x = x^\star$

then $x^\star$ is globally **asymptotically** stable.

– $\dot{V}$ is not positive definite

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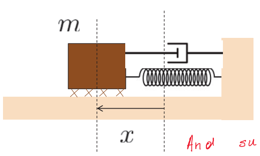
Example (revisited)

$$m\ddot{x} = -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3 \quad (*)$$



$$V(x, \dot{x}) = (\underbrace{2m\dot{x}^2}_{\text{kinetic}} + \underbrace{2k_0 x^2 + k_1 x^4}_{\text{potential energy}})/4 > 0, \qquad V(0,0) = 0$$

$\dot{V}(x, \dot{x}) = -b|\dot{x}|^3$ gives $E = \{(x, \dot{x}) : \dot{x} = 0\}$.

- $\dot{V} \leq 0$ *(it is not negative definite because it can be zero for all x)*
  negative semidefinite
- $\dot{V} \equiv 0 \Rightarrow \dot{x} \equiv 0 \overset{\ddot{x}=0}{\Longrightarrow} k_0 x(1 + \frac{k_1}{k_0}x^2) \equiv 0 \Rightarrow x_1 = 0$

$\dot{V} \equiv 0$ implies $\dot{x} \equiv 0$
Differentiating we get $\ddot{x} \equiv 0$.
And substituting in $(*)$ we get $x \equiv 0$

Barbashin, Krasovskii or LaSalle

- Global asymptotic stability of $(x, \dot{x}) = (0, 0)$

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Exponential stability

Theorem: Let $\dot{x} = f(x)$ and $f(x^\star) = 0$. If there exists a $\mathcal{C}^1$ function $V : \Re^n \to \Re$ and numbers $\alpha, \epsilon, c > 0$ such that

More strict condition than p.d.

(1) $V(x^\star) = 0$

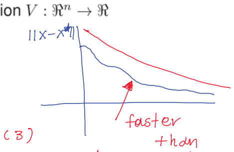
(2) $V(x) > \epsilon\|x - x^\star\|^c > 0$ for all $x \neq x^*$

(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

then $x^\star$ is globally **exponentially** stable.

radially unbounded



From (3)

$V(x) \leq V(0)\, e^{-\alpha t}$

From (2)

$\epsilon \|x - x^\star\|^c \leq V(0)\, e^{-\alpha t}$

$\Rightarrow \|x - x^\star\| \leq \left(\frac{V(0)}{\epsilon}\right)^{\frac{1}{c}} e^{-\frac{\alpha}{c} t}$

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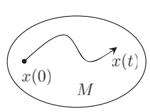
Invariants Sets

Invariant set M for the system $\dot{x} = f(x)$.

$$x(0) \in M \Longrightarrow x(t) \in M \ \forall t \geq 0.$$



Examples:
- equilibrium points
- limit cycles
- the whole $\Re^n$
- Lyapunov sets

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Lyapunov sets as invariant sets

- Notice that the condition $\dot{V} \leq 0$ implies that if a trajectory crosses a Lyapunov surface $V(x) = \gamma$ it can never come out again.

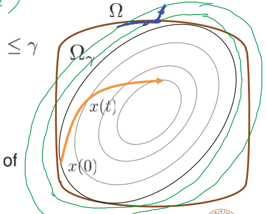
see next slides for explanation on why 0 is not invariant

$\dot{f} \leq 0 \Rightarrow f(t_2) \leq f(t_1)$

Why? $\boxed{x(0) \in \Omega_\gamma} \quad \dot{V} \leq 0 \Longrightarrow \quad V(x(t)) \leq V(x(0)) \leq \gamma$

$V(x(0)) \leq \gamma$

$$x(t) \in \Omega_\gamma = \{x \in \Re : \ V(x) \leq \gamma\}, \forall t \geq 0$$



- If $V(x) \in \mathcal{C}^1$ and satisfies $\dot{V}(x) \leq 0$ along the solutions of $\dot{x} = f(x)$, then the set:

$$\Omega_\gamma = \{x \in \Re : \ V(x) \leq \gamma\} \subset \Omega$$

is an invariant set.

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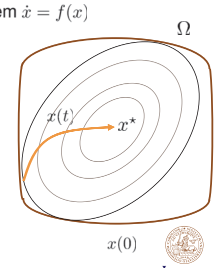
Region of Attraction

- Local asymptotic stability theorems guarantee existence of a possibly small neighborhood of the equilibrium point where such an attraction takes place
- The region of attraction to the equilibrium point $x^\star$ of the system $\dot{x} = f(x)$ is defined by $\mathcal{R}_A = \{x(0) \in \Omega : x(t) \to x^\star \text{ as } t \to \infty\}$.
- If $V(x) \in \mathcal{C}^1$ and satisfies $\dot{V}(x) \le 0$ along the solutions of $\dot{x} = f(x)$, then the set:

$$\Omega_\gamma = \{x \in \Re : \ V(x) \le \gamma\} \subset \Omega$$

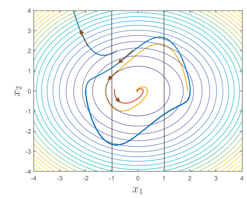
is an invariant set and can be used as an **estimate of region of attraction**.

- The estimate of the region of attraction based on Lyapunov level sets is conservative $\Omega_\gamma \subset \mathcal{R}_A$



Discussion

Why we cannot claim that Ω is an estimate of region of attraction?



$\Omega = \{x \in \Re^n : \quad \dot{V} \le 0\}$

Van der Pol equation in reverse time

$\dot{x}_1 = -x_2$

$\dot{x}_1 = x_1 - (1 - x_1^2)x_2$

$V = \frac{1}{2}(x_1^2 + x_2^2)$ positive definite for all x

$\dot{V} = -(1 - x_1^2)x_2^2$ negative semidefinite for $|x_2| < 1$

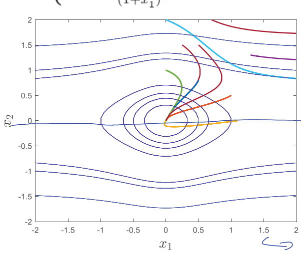
The conditions for applying LaSalle Theorem for asymptotic stability are satisfied in $\Omega = \{|x_1| < 1, |x_2| < L\}$ L arbitrarily large constant.

Thus, the origin is local asymptotically stable. However Ω is not invariant. Starting within Ω the trajectory can move to Lyapunov surfaces $V(x) = \gamma$ with smaller γs but there is no guarantee that the trajectory will remain in Ω. See the blue trajectories. Once leaving Ω, $\dot{V}$ could be positive and the trajectory may move to Lyapunov surfaces with higher γ. Observe that one of the blue trajectories is a limit cycle. Charecterize its stability.

Example 1

Example:

$$\begin{cases} \dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{cases}$$



$\begin{cases} x_2 = 0 \\ \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2 = c \end{cases} \Rightarrow \left\{ \frac{1}{2}\frac{x_1^2}{1+x_1^2} = c \right.$

$\Rightarrow (1-2c)\,x_1^2 = 2c$

It has solution when $c < \frac{1}{2}$ (*)

- $V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2 > 0$ and $V(0) = 0$
- $V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$ is radially unbounded for $V(x) < 1$
- $\dot{V} = \frac{x_1\dot{x}_1}{(1+x_1^2)^2} + x_2\dot{x}_2 = -6\frac{x_1^2}{(1+x_1^2)^4} - 2\frac{x_2^2}{(1+x_1^2)^2} < 0, \dot{V}(0) = 0$

$x_2 = 0$

The axis $x_2 = 0$ intersects with a Lyapunov surface $V(x) = c$ when $c < \frac{1}{2}$

$\Rightarrow V(x) = c$ is closed

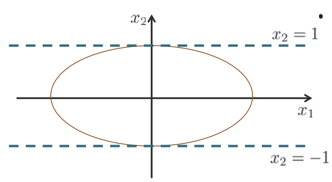
- Local asymptotic stability
- Estimate of region of attraction $\Omega_{1/2} = x \in \Re^2 : V(x) < 1/2$

Example 2

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = \frac{x_1}{2} + x_2^3 - x_2 \end{cases}$$

Eq. Point

$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases}$



- $V = \frac{x_1^2}{2} + x_2^2 > 0$ and $V(0) = 0$
- $V = \frac{x_1^2}{2} + x_2^2$ radially unbounded
- $\dot{V} = x_1\dot{x}_1 + 2x_2\dot{x}_2 = \cdots = 2x_2^2(x_2^2 - 1)$
  - $\dot{V} \le 0$ for $|x_2| < 1$
  - $\dot{V} \equiv 0 \Rightarrow x_2 \equiv 0 \overset{\dot{x}=f(x)}{\Longrightarrow} x_1 \equiv 0$ for $|x_2| < 1$

$\dot{V} = \nabla V \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$

$= x_1\dot{x}_1 + 2x_2\dot{x}_2$

$= -\cancel{x_1x_2} - 2x_2^2 + 2x_2^4 + \cancel{x_1x_2}$

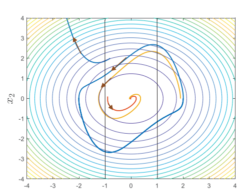
$\dot{V} \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$

$V(x) < 1$

LaSalle

- Local asymptotic stability
- Estimate of region of attraction $\Omega_1 = x \in \Re^2 : V(x) < 1$

Example 3



Van der Pol equation in reverse time

$\dot{x}_1 = -x_2$

$\dot{x}_1 = x_1 - (1 - x_1^2)x_2$

$V = \frac{1}{2}(x_1^2 + x_2^2)$ positive definite for all x

$\dot{V} = -(1 - x_1^2)x_2^2$ negative semidefinite for $|x_1| < 1$

The conditions for applying LaSalle Theorem for asymptotic stability are satisfied in $\Omega = \{|x_1| < 1, |x_2| < L\}$ L arbitrarily large constant. Thus, the origin is local asymptotically stable.

- Derive an estimate of the region of attraction.
- Which is the actual region of attraction?

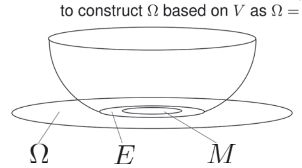
LaSalle's invariance principle

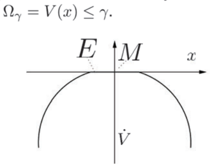
LaSalle's invariant set Theorem

- Let $\Omega \subseteq \Re^n$ compact invariant set for $\dot{x} = f(x)$.
- Let $V : \Omega \to \Re$ be a $\mathcal{C}^1$ function such that $\dot{V}(x) \le 0, \ \forall x \in \Omega$.
- $E := \{x \in \Omega : \ \dot{V}(x) = 0\}$, M :=largest invariant subset of E

$\forall x(0) \in \Omega$, $x(t)$ approaches M as $t \to +\infty$

Note that Ω can be defined independent of V. In many cases, it is easier to construct Ω based on V as $\Omega = \Omega_\gamma = V(x) \le \gamma$.



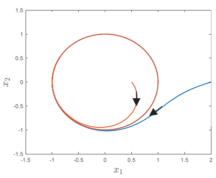


Example – Limit Cycle

Show that $M = \{x : \|x\| = 1\}$ is a asymptotically stable limit cycle for (almost globally, except for starting at $x = 0$)

$\|x\|^2 = x_1^2 + x_2^2.$

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$

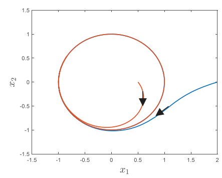


The system has one equilibrium at the origin and one limit cycle. Thus the set of trajectories that are invariant for the system are in the set $E = \{x : \|x\| = 0 \text{ or } \|x\| = 1\}$. We can actually show this by calculating the derivative $\frac{d}{dt}(\|x\|^2) = -\|x\|^2(\|x\|^2 - 1)$:

- If $\|x(0)\| = 0$ or $\|x(0)\| = 1$ the derivative is zero $\frac{d}{dt}(\|x(0)\|^2)$and thus the norm of x will not change.
- $\|x(t)\| = 0$ corresponds to the equilibrium point at the origin $\dot{x}_1 = \dot{x}_2$ while $\|x(t)\| = 1$ corresponds to $\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -x_1 \end{cases}$ defining a limit cycle moving clockwise
- Remark: From the derivative $\frac{d}{dt}(\|x\|^2) = -2\|x\|^2(\|x\|^2 - 1)$ and if we consider a Lyapunov-like function $V_0 = \frac{1}{2}$ we can see that $\Omega' = \{\|x\| < 1\}$ cannot be proved invariant since $\dot{V}_0 > 0$.

Example – Limit Cycle

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$



- Take the Lyapunov-like $V = (x_1^2 + x_2^2 - 1)^2$, that is positive but not positive definite. It encodes some dinstance metric from the limit cycle $x_1^2 + x_2^2 = 1$.
- Differentiating V along the system trajectories, we get: $\dot{V} = -4(x_1^2+x_2^2)(x_1^2+x_2^2-1)^2 \leq 0$.
- Choose $\Omega = \{x \in \Re^2 : 0 < \|x\| \leq 1\}$ to exclude $\|x\| = 0$. Note that Ω is invariant as it is subset of $\Omega_1 = \{V < 1\}$. Check this. By excluding $x = 0$, the maximum invariant set is $M = \{x \in \Omega : \|x\| = 1\}$.

LaSalle's Invariance Principle

$x \to M$ as $t \to \infty$ *almost globally*

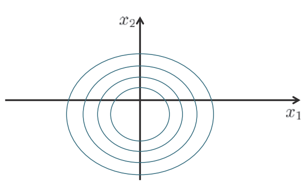
Example – Set of equilibriums

$$\begin{cases} \dot{x}_1 = \lambda x_1 - x_1 x_2 \\ \dot{x}_2 = a x_1^2 \end{cases}$$

λ constant

At equilibrium

$$\begin{cases} \dot{x}_1 = \lambda x_1 - x_1 x_2 = 0 \\ \dot{x}_2 = a x_1^2 = 0 \end{cases} \implies x_1 = 0,\ x_2 \in \Re$$



c constant

- $V = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - c)^2 > 0$, radially unbounded
- $\dot{V} = -(c - \lambda)x_1^2 \leq 0$ for $c > \lambda$
- $E := \{x \in \Re^2 : \dot{V}(x) = 0\} \equiv M := \{x \in \Re^2 : x_1 = 0\}$

LaSalle's Invariance Theorem

- $x \to M$ as $t \to \infty$

Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

$$\dot{x} = Ax \quad = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

Eigenvalues of $A : \{-1, -3\}$

$\Rightarrow$ (global) asymptotic stability.

Try to prove stability with:

$V(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2}x^T x = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$

$= -x_1^2 + 4x_1x_2 - 3x_2^2$

$= -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det(M) = 3 - 4 = -1$

$-\dot{V}$ is not p.d.

Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

Try to prove stability with:

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$

$V(x) = \|x\|^2 = x^T P x$ *Parametric Lyapunov function in a quadratic form*

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T(PA + A^T P)x$ *Choose parameters for P such that* $-\dot{V}(x)$ *p.d.*

$-x^T Q x < 0$

$\hookrightarrow$ e.g. $Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$ $q_1, q_2 > 0$

Lyapunov analysis for Linear systems

1. Let $Q = I_2$
2. Solve P from the Lyapunov equation

$$\dot{x} = Ax \quad = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for p_{11}, p_{12} and p_{22} gives

$$\begin{aligned} 2p_{11} &= -1 \\ -4p_{12} + 4p_{11} &= 0 \\ 8p_{12} - 6p_{22} &= -1 \end{aligned} \implies \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

or

1. Choose an arbitrary symmetric, positive definite matrix Q.
2. Find P that satisfies Lyapunov equation

$$PA + A^T P = -Q$$

and verify that it is positive definite.

Lyapunov function: $V(x) = x^T P x$

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = -x^T Q x$

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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 7: Indirect Lyapunov's method and Input-Output Stability

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AUTOMATIC CONTROL, FACULTY OF ENGINEERING.
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– Lab 1 today
– Lecture 28 Nov Monday
→ 2 Dec Friday

Outline

- Lyapunov Analysis for Linearized systems
- Indirect Lyapunov's Method
- Small-gain theorem
- Circle Criterion (the point -1/k is replaced by a cycle)

Indirect Lyapunov method.

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Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

or

1. Choose an arbitrary symmetric, positive definite matrix Q.
2. Find P that satisfies Lyapunov equation

$Ax = b$ $\rightarrow$ $\boxed{PA + A^T P = -Q}$

and verify that it is positive definite.

Lyapunov function: $\boxed{V(x) = x^T P x}$

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = -x^T Q x$

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Lyapunov analysis for Linear systems

1. Let $Q = I_2$
2. Solve P from the Lyapunov equation

$\lambda_{min}(M)\|x\|^2 \leq \underbrace{x^T M x} \leq \lambda_{max}(M)\|x\|^2$

$$\dot{x} = Ax = \underbrace{\begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for p_{11}, p_{12} and p_{22} gives

$\dot{V} = -x^T Q x$

$\dot{V} \leq \underbrace{-\lambda_{min}(Q)\|x\|^2}$

$$2p_{11} = -1$$
$$-4p_{12} + 4p_{11} = 0$$
$$8p_{12} - 6p_{22} = -1$$

$$\Longrightarrow \overset{P}{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

$\boxed{\lambda_{min}(P)\|x\|^2 \leq V \leq \lambda_{max}(P)\|x\|^2}$ | Exponential stability theorem.

$\det(P) > 0$

p_{11} or $p_{22} > 0$

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Lyapunov's indirect method

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $f(0) = 0$. Linearization

$$\dot{x} = Ax + g(x), \qquad ||g(x)|| = o(||x||) \text{ as } x \to 0.$$

(1) Re $\lambda_k(A) < 0, \forall k \quad \Rightarrow \quad x = 0$ locally asympt. stable

(2) $\exists k :$ Re$\lambda_k(A) > 0 \quad \Rightarrow \quad x = 0$ unstable

Lyapunov function candidate: $V(x) = x^T P x$

Jacobian

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\dot{x} = \underbrace{\frac{\partial f}{\partial x}\Big|_{x=0}}_{\Downarrow\ A} x + \underbrace{\left[f(x) - \frac{\partial f}{\partial x}\Big|_{x=0} x\right]}_{\Downarrow\ g(x)}$$

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} \qquad A^T P + PA = -Q$$

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Lyapunov's indirect method

– Choose Q and solve $PA+A^TP=-Q$.

Lyapunov function candidate: $V(x) = x^T P x$

Differentiating $\dot V(x)$ *along system's trajectories* $\dot x = Ax + g(x) = f(x)$

nonlinear part

$\dot V = \dot x^T P x + x^T P \dot x$

$$\dot V(x) = x^T P f(x) + f^T(x) P x$$
$$= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px$$
$$= x^T(PA + A^T P)x + 2x^T Pg(x) = -x^T Qx + \boxed{2x^T Pg(x)}$$

$-Q$

$\|\alpha^T b\| \leq \|\alpha\|\|b\|$

$$\boxed{x^T Qx \geq \lambda_{\min}(Q)\|x\|^2}$$

$\dot V \leq -\lambda_{min}(Q)\|x\|^2 + 2\|x\| \, \|P\| \, \|g(x)\|$

and for all $\gamma > 0$ there exists $r > 0$ such that

$\dot V$ negative

$$\boxed{\|g(x)\| < \gamma\|x\|, \quad \forall\|x\| < r}$$

$\dot V \leq -\theta\|x\|^2$

Thus, choosing γ sufficiently small gives

$$\boxed{\dot V(x) \leq -(\lambda_{\min}(Q) - 2\gamma\lambda_{\max}(P))\|x\|^2 < 0}$$

$\lambda_{m}(Q) - 2\gamma\lambda_{max}P > 0$

$\gamma < \frac{\lambda_{min}Q}{2\lambda_{max}P}$

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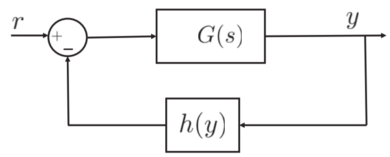
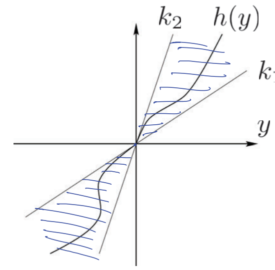
Feedback form where the nonlinearity is in a block





$\dot x = Ax + g(x) = f(x)$

$= Ax + B\, g(x)$

I

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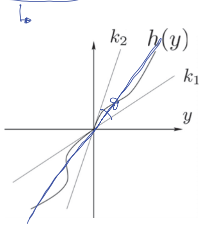
Sector nonlinearity

$\dot y = h(y)$



Gain

y $h(y)$

Bound in sector function $h \in \text{sector}[k_1, k_2]$

- $h(y)$ continuous wrt y
- $h(0) = 0$
- $k_1 \leq \frac{h(y)}{y} \leq k_2, \forall y \neq 0$ or equivalently $\boxed{k_1 y^2 \leq yh(y) \leq k_2 y^2 \ \forall\ y}$

inequality →]

Other cases:

- $h \in \text{sector}(k_1, k_2]$ ⇒ $k_1 < \frac{h(y)}{y} \leq k_2$
- $h \in \text{sector}[k_1, \infty)$ ⇒ $k_1 \leq \frac{h(y)}{y}$
- $h \in \text{sector}[0, \infty)$ (first and third quadrant)

$h \in [0, \infty)$

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Signal norms and spaces

- A signal $x(t)$ is a function from $\mathbf{R}^+$ to $\mathbf{R}^d$. $x(t) \in \mathbb{R}^d$ $\|x\| = \sqrt{x_1^2 + \ldots + x_d^2}$
- A signal norm is a way to measure the size of $x(t)$ in long run:

2-norm (energy norm): $\|x\|_2 = \sqrt{\int_0^\infty \|x(t)\|^2 dt}$

sup-norm: $\|x\|_\infty = \sup_{t \in \mathbf{R}^+} |x(t)|$

- The space of signals with $\|x\|_2 < \infty$ is denoted $\mathcal{L}_2$. $x(t) \in \mathcal{L}_2$
- The space of signals with $\|x\|_\infty < \infty$ is denoted $\mathcal{L}_\infty$.
- $x(t) \in \mathcal{L}_2$ corresponds to bounded energy signals.
- $x(t) \in \mathcal{L}_\infty$ corresponds to bounded signals.

Equivalent expression in frequency domain

$$\|x\|_2^2 = \int_0^\infty \|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|X(j\omega)\|^2 d\omega$$

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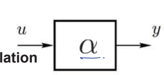
Gain of a system $\mathcal{L}_2$

Gain of S: $\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$ $u \rightarrow \boxed{S} \rightarrow y$

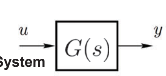
S is bounded-input bounded-output (BIBO) stable if $\gamma(S) < \infty$.

Static linear relation



$$\gamma(\alpha) = \sup_{u \in \mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{|\alpha|\|u\|_2}{\|u\|_2} = |\alpha|$$

Stable Linear System

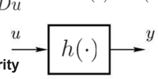


$\dot x = Ax + Bu$
$y = Cx + Du$

$G(s) = C(sI - A)^{-1}B + D$

$$\gamma(G) = \sup_{u \in \mathcal{L}_2} \frac{\overset{y=}{\|Gu\|_2}}{\|u\|_2} = \sup_{\omega \in (0,\infty)} |G(j\omega)|$$

Static Nonlinearity



$h(\cdot) \in [-K \quad K]$

$$\boxed{\gamma(h) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = K}$$

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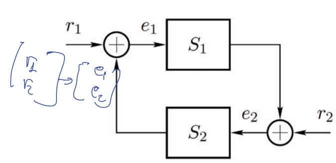
The Small-Gain Theorem



$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$\gamma(S_1) < \infty$, $\gamma(S_2) < \infty$

Theorem

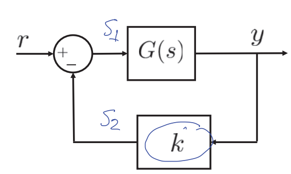
Assume S_1 and S_2 are BIBO stable. If

$$\gamma(S_1)\gamma(S_2) < 1$$

then the closed-loop map from (r_1, r_2) to (e_1, e_2) is BIBO stable.

$\gamma(G(s)) = \sup_{\omega \in (0,\infty)} |G(j\omega)|$

$\sup_{\omega \in (0,\infty)} |G(j\omega)| \ K < 1.$



$$\sup_{\omega \in (0,\infty)} |G(j\omega)| < \frac{1}{k}$$

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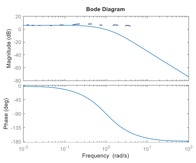
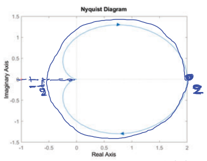
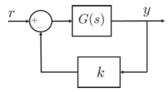
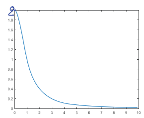
Small-Gain Theorem is conservative







$G(s) = \frac{2}{(s+1)^2}$



Small gain Theorem $\rightarrow G(s) = \frac{2}{s^2+2s+1}$

$\gamma_a \cdot \gamma_k < 1$

$\sup_{\omega\in(0,\infty)} |G(j\omega)||k| < 1$

$\boxed{|k| < \frac{1}{2}}$

$\rightarrow G(j\omega) = \frac{2}{1-\omega^2+2j\omega} = \frac{2(1-\omega^2)}{(1-\omega^2)^2+4\omega^2} - j\frac{4\omega}{(1-\omega^2)^2+4\omega^2}$

$\boxed{|G(j\omega)| = \frac{2}{\sqrt{(1-\omega^2)^2+4\omega^2}}}$

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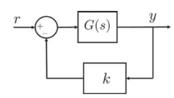
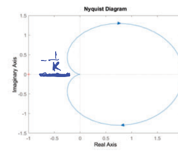
Nyquist criterion





- $P_{CL_{Re>0}} = N(-1/k, 0) + P_{OL_{Re>0}}$

- Given a stable open loop system, the closed loop is stable if the Nyquist plot of the open loop system does not encircle the point (-1/k,0) in the clockwise direction.

$k_1 \leq \frac{h(y)}{y} \leq k_2 \Rightarrow k_1 = k_2$

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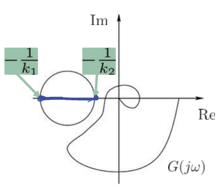
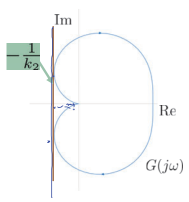
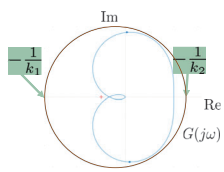
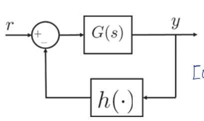
Circle criterion









$h(x) \in [k_1, k_2]$

- $G(s)$ **stable**
  - $0 < k_1 < k_2$: The Nyquist curve of $G(s)$ does not intersect or encircle the circle defined by the points $-1/k_1$ and $-1/k_2$
  - $[0, k_2]$ $0 = k_1 < k_2$: The Nyquist curve of $G(s)$ stays to the right of the line Re $s = -1/k_2$
  - $k_1 < 0 < k_2$: The Nyquist curve of $G(s)$ stays inside the circle

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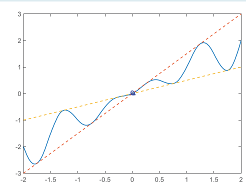
Circle criterion – Example 1



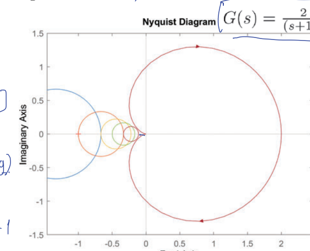
$$\begin{cases} \dot{x} = -x + r - h(y) \\ \dot{y} = -y + 2x \end{cases} \Rightarrow$$

$sX = -X + r - h(y)$

$sY = -Y + 2X \Rightarrow (s+1)Y = 2X$

$h(y) = y + \frac{1}{2}y\sin(y)$ $(s+1)X = r - h(y)$

$\boxed{G(s) = \frac{2}{(s+1)^2}}$



1) Find the sector

$h(y) \in [0.5, 1.5]$

$h(y) = y + \frac{1}{2}\sin(y)\, y \Rightarrow h(0) = 0$

$\Rightarrow \frac{h(y)}{y} = 1 + \frac{1}{2}\sin(y)$

2) Apply Circle Criterion depending on the sector

$= 1 + \frac{1}{2}\sin(y)$

$1/2 + 1/2 \quad 1/2 + 1$

$1 + -1 \leq \sin y \leq 1$

$0.5 \leq h(y) \leq 1.5$

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Circle criterion – Example 2

$$\begin{cases} \dot{x} = -x + r - f(y,t) \\ \dot{y} = -y + 2x \end{cases}$$

$\boxed{f(y,t) = (3 - 4e^{-2t})y}$

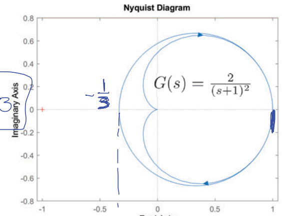
1) Find the sector

$f(0,t) = 0$

$\frac{f(y,t)}{y} = 3 - 4e^{-2t}$

$\boxed{-1 \leq \frac{f(y,t)}{y} \leq 3}$

2) Apply Circle Criterion depending on the sector



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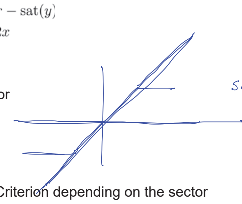
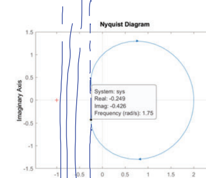
Circle criterion – Example 3

$$\begin{cases} \dot{x} = -x + r - \mathrm{sat}(y) \\ \dot{y} = -y + 2x \end{cases}$$

1) Find the sector



sector $[0, 1]$



2) Apply Circle Criterion depending on the sector

$G(j\omega) = \frac{2}{1-\omega^2+2j\omega} = \frac{2(1-\omega^2)}{(1-\omega^2)^2+4\omega^2} - j\frac{4\omega}{(1-\omega^2)^2+4\omega^2}$

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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 8: Input-Output Stability
Intro to Control-design

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Outline

- Circle criterion and positive real functions (passivity)
- Control design based on linearization
- Lyapunov-based control design

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(Strictly) Positive Real Transfer Functions

closed $[0,1]$ open $(0,1)$

A proper rational transfer function ~~matrix~~ $G(s)$ is positive real if:

- The poles of $G(s)$ are in $\mathrm{Re} \leq 0$
- $\mathrm{Re}[G(j\omega)] \geq 0, \forall \omega \in [0, \infty) \longrightarrow$ Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane.

positive real

$G(s)$ is called strictly positive real if $G(s-\varepsilon)$ is positive real for some $\varepsilon > 0$
It is easier to check directly the following conditions:

- The poles of $G(s)$ are in $\mathrm{Re} < 0$.
- $\mathrm{Re}[G(j\omega)] > 0, \forall \omega \in [0, \infty)$ and $G(\infty) \geq 0$ or $\boxed{\lim_{\omega\to\infty} \omega^2 \mathrm{Re}[G(j\omega)] > 0}$ $\longrightarrow$ Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane and does not touch the Imaginery axis.

strictly positive real

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Quiz

- $G(s) = \frac{1}{s}$ $\Rightarrow$ $G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} \Rightarrow \mathrm{Re}(G(j\omega)) = 0$
- $G(s) = \frac{1}{s+1}$ $\Rightarrow$ $G(j\omega) = \frac{1}{j\omega+1} = \frac{1-j\omega}{1+\omega^2} \Rightarrow \mathrm{Re}[G(j\omega)] = \frac{1}{1+\omega^2} > 0$
- $G(s) = \frac{1}{s^2+2s+1}$

$G(j\omega) = \frac{1}{-\omega^2+2j\omega+1} = \frac{1}{(1-\omega^2)+2j\omega} = \frac{(1-\omega^2)-2j\omega}{(1-\omega^2)^2+4\omega^2}$ for $|\omega|>1$ $\mathrm{Re}(j\omega)<0$ Not positive real

$= \frac{1}{\omega^2\left(\frac{1}{\omega^2}+1\right)}$ $G(\infty) = \omega^2 \frac{1}{\omega^2\left(\frac{1}{\omega^2}+1\right)} \to 0$

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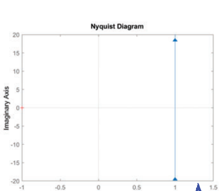
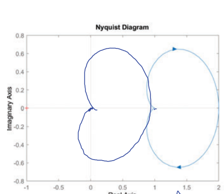
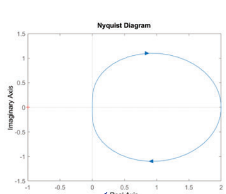
Matching Quiz







a. $G(s) = \frac{s+1}{s} = 1 + \frac{1}{s}$ (integrator + 1)

b. $G(s) = \frac{s+2}{(s+1)^2}$

c. $G(s) = \frac{s^2+2s+2}{s^2+2s+1} = \frac{1+(s+1)^2}{(s+1)^2} = \boxed{1 + \frac{1}{(s+1)^2}} = 1 + $ (Non positive real transfer function)

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Kalman Yakubovich Popov Lemma

$G(s) = C(sI-A)^{-1}B + D$

Minimal realization of $G(s)$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- (A, B): controllable
- (A, C): observable

$$R = [B \quad AB \cdots A^{n-1}B]$$

$$S = \begin{bmatrix} C \\ AC \\ \vdots \\ A^{n-1}C \end{bmatrix}$$

$G(s)$ strictly positive real $\quad \exists P = P^T > 0, Q = Q^T \geq 0, \epsilon > 0$ s.t. $\begin{cases} PA + A^TP &= -Q - \epsilon P \\ PB &= C^T \end{cases}$

$G(s)$ positive real $\quad \exists P = P^T > 0, Q = Q^T \geq 0$ s.t. $\begin{cases} PA + A^TP &= -Q \\ PB &= C^T \end{cases}$

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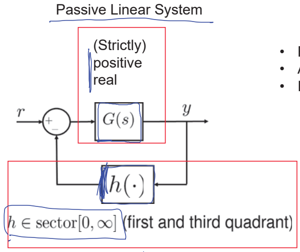
Passivity Theorem(s)

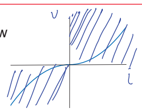


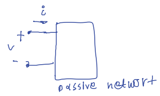
- Feedback interconnection of passive systems is passive
- A passive system is BIBO
- If the input r=0, the origin of minimum realization of G(s) is gas

$h \in [0, \infty]$ and $Z(s) = G(s)$ is strictly positive real

The feedback system is BIBO

Power inflow
$p = vi$
$p = Fv$



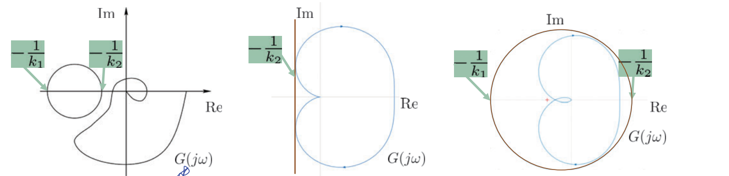


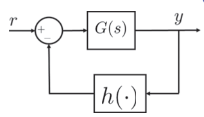
$p(t) = v(t)\, i(t)$

$E(t) = E(0) + \int_0^t p(t)$
energy absorbed

if $p(t) > 0$ box absorbs energy
if $p(t) < 0$ box delivers energy

Passivity theorem – Circle criterion





$0 < k_1 < k_2$

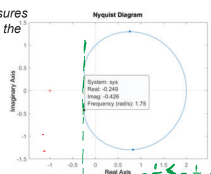
- $h \in [0, \infty]$ and $Z(s) = G(s)$ is strictly positive real
- $h \in [k_1, \infty]$ and $Z(s) = \frac{G(s)}{1 + k_1 G(s)}$ is strictly positive real
- $h \in [k_1, k_2]$ and $Z(s) = \frac{1 + k_2 G(s)}{1 + k_1 G(s)}$ is strictly positive real

The system is BIBO and if $r = 0$ the origin GAS

Circle criterion – Strictly positive real functions

$$\begin{cases} \dot{x}_1 = -x_1 - \operatorname{sign}(x_1)\min(K|x_1|, 1) \\ \dot{x}_2 = -x_2 + 2x_1 \end{cases}$$

Find the maximum value of K that ensures asymptotic stability of the origin using the circle criterion.



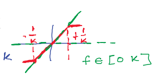
$-0.25 \quad -\frac{1}{k} < -\frac{1}{4} \Rightarrow$

1) Isolate the linear part of the system (A). Check the argument (input) of the nonlinear function (c). Check how the nonlinearity is involved in the system(b). Find G(s) using A, b, c.

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases} \qquad A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad c = [1 \;\; 0]$$

2) Nonlinearity: Find the sector as a function of the gain K

$\operatorname{sign}(x_1)\min(K|x_1|, 1)$ if $K|x_1| < 1$ $f(x) = Kx_1$; if $K|x_1| > 1$ $f(x) = \operatorname{sgn}(x_1)$ k



$f \in [0\; K]$

if $\Delta < 0$ then the quadratic polynomial is positive for any $x = \omega^2$

3) Check is the Z(s) is strictly positive real

$$G(j\omega) = \frac{2}{1 - \omega^2 + 2j\omega} = \frac{2(1-\omega^2)}{(1-\omega^2)^2 + 4\omega^2} - j\frac{4\omega}{(1-\omega^2)^2 + 4\omega^2}$$

Check the discriminant (Δ) of the quadratic equation $x^2 + 2(1-k)x + 2K + 1 = 0$

$$\operatorname{Re}\left[1 + KG(j\omega)\right] = 1 + K\frac{2(1-\omega^2)}{(1-\omega^2)^2 + 4\omega^2} = \frac{\omega^4 + 2(1-K)\omega^2 + 2K + 1}{(1-\omega^2)^2 + 4\omega^2} \Rightarrow K < 4$$

$\Delta = 2(1-k)^2 - 4(2k+1)$ (*)

Linear control design based on linearization

$\dot{x} = f(x, u)$

$$\frac{d\tilde{x}}{dt} = A\tilde{x} + B\tilde{u}$$

$$A = \frac{\partial f}{\partial x}(x^\star, u^\star) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=x^\star, u=u^\star}$$

$$B = \frac{\partial f}{\partial u}(x^\star, u^\star) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\right|_{x=x^\star, u=u^\star}$$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(A, B) is not controllable

Controllability condition: $\operatorname{rank}(R) = n$

$$R = [B \quad AB \cdots A^{n-1}B]$$

$\dot{x} = Ax + Bu$
$u = -Kx$
CLS $= (A - BK)x$

$$\dot{x} = f(x, u^\star - K(x - x^\star)) \qquad u = u^\star - K(x - x^\star)$$

$$\dot{\tilde{x}} = (A - BK)\tilde{x}$$

Example (Linearization)

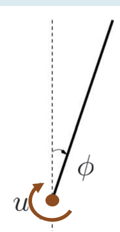
An inverted pendulum controlled by a motor torque u at the joint:

$$\ddot{\phi}(t) = \frac{g}{l}\sin(\phi(t)) + \frac{1}{ml^2}u,$$

where $u(t)$ is acceleration, can be written as

$x^\star = \begin{bmatrix} \delta \\ 0 \end{bmatrix} \begin{matrix} \phi \\ \dot{\phi} \end{matrix}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g}{l}\sin(x_1) - \frac{1}{ml^2}u \equiv 0 \end{aligned} \qquad u^\star = mgl\sin(\delta)$$



Linearize the system and find a control input that can stabilize the system at angle δ? Is the linear system controllable for all δ?

$$A = \left.\begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(x_1) & 0 \end{bmatrix}\right|_{\substack{x=x^\star \\ u=u^\star}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\delta & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & -\frac{1}{ml^2} \\ -\frac{1}{ml^2} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\delta & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{ml^2} \\ 0 \end{bmatrix}$$

$$u = mgl\sin\delta - k_1\dot{\phi} - k_2(\phi - \delta)$$

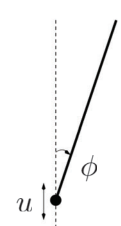
Example (Linearization)

An inverted pendulum with vertically moving pivot point

$$\ddot{\phi}(t) = \frac{1}{l}\left(g + u(t)\right)\sin(\phi(t)),$$

where $u(t)$ is acceleration, can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{l}(g + u)\sin(x_1) \end{aligned}$$



Try this home – See lecture notes 2.

Lyapunov-based design

Steps of Lyapunov-based design:

1. Select a positive definite $V(x)$. $\dot{x} = f(x, u)$
2. Calculate $\dot{V}(x) = \boxed{\frac{\partial V}{\partial x}} f(x, u)$.
3. Find a (possibly) nonlinear feedback control law that makes $\dot{V}$ negative.

$$\boxed{\frac{\partial V}{\partial x} f(x, u) \leftarrow 0}$$

- $\dot{V} \leq 0 \longrightarrow x = 0$ may be asymptotically stable (check LaSalle)
- $\dot{V} < 0$ for all $x \neq 0 \longrightarrow x = 0$ asymptotically stable
- $\dot{V} \leq -\lambda V \longrightarrow x = 0$ exponentially stable if addtionally $V \geq c\|x\|^2$

Comments:

- Selection of $V(x)$
- Depends on the system dynamics $\dot{x} = f(x, u)$

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Example 1 (Lyapunov-based design)

Consider the nonlinear system

$$\dot{x}_1 = -3x_1 + 2x_1 x_2^2 + u$$
$$\dot{x}_2 = -x_2^3 - x_2,$$

Find a nonlinear feedback control law which makes the origin globally asymptotically stable.

$\dot{V} = -3x_1^2 - x_2^2 - x_2^4$

$\dot{V} = -x_1^2 - x_2^2 - 2x_1^2 - x_2^4$

$\dot{V} \leq -x_1^2 - x_2^2 = -2V \Rightarrow \boxed{\dot{V} \leq -2V}$

$V = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V} = [x_1 \quad x_2] \begin{bmatrix} -3x_1 + 2x_1 x_2^2 + u \\ -x_2^3 - x_2 \end{bmatrix} = \underbrace{-3x_1^2}_{<} + \underbrace{2x_1^2 x_2^2}_{>0} + x_1 u \underbrace{- x_2^4}_{<0} \underbrace{- x_2^2}_{<0}$

$u = -2x_1 x_2^2$

$= -3x_1^2 + \cancel{2x_1^2 x_2^2} - \cancel{2x_1^2 x_2^2} - x_2^4 - x_2^2$

$\dot{V} < 0$ for all $x \neq 0$

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Example 2 (Lyapunov-based design)

Consider the nonlinear system

$$\dot{x}_1 = -3x_1 + 2x_1 x_2^2 + u$$
$$\dot{x}_2 = -x_2^3 - x_2,$$

Find a nonlinear feedback control law which makes the origin globally exponentially stable.

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Example 3 (Lyapunov-based design)

Consider the system

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = u$$

Find a globally asymptotically stabilizing control law $u = u(x)$.

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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 9: Control design for nonlinear systems

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Outline

- Lyapunov-based control design
- Exact feedback linearization

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Lyapunov-based design

$\dot{x} = f(x,u)$

Steps of Lyapunov-based design:

1. Select a positive definite $V(x)$.
2. Calculate $\dot{V}(x) = \frac{\partial V}{\partial x} f(x, u)$.
3. Find a (possibly) nonlinear feedback control law that makes $\dot{V}$ negative.

- $\dot{V} \leq 0 \longrightarrow x = 0$ may be asymptotically stable (check LaSalle)
- $\dot{V} < 0$ for all $x \neq 0 \longrightarrow x = 0$ asymptotically stable
- $\dot{V} \leq -\lambda V \longrightarrow x = 0$ exponentially stable if addtionally $V \geq c\|x\|^2$

Comments:

- Selection of $V(x)$
- Depends on the system dynamics $\dot{x} = f(x, u)$

Example 3 (Lyapunov-based design)

Consider the system

$V = \frac{1}{2}(x_1 - x_1^*)^2 + \frac{1}{2}(x_2 - x_2^*)^2$

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = u$$

$\dot{x} = f(x,u) = \begin{bmatrix} x_2^3 \\ u \end{bmatrix}$

For (*) $\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -x_1 - kx_2 \end{cases}$

Find a globally asymptotically stabilizing control law $u = u(x)$.

$x^* = 0$

1. $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ p.d. $x^* = 0$
2. $\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} = x_1 x_2^3 + x_2 u$

$u = -x_1 x_2^2 - \cancel{x_1} x_2$

$\dot{V} = -x_1^2 x_2^2$

$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = \cancel{x_1^2 x_2^2} - x_1 x_2^2 - x_1^2 x_2 \end{cases}$

Correction

$E = \{\dot{V} = 0,\ x_1^2 x_2^2 = 0\}$.

Note the equilibrium is $(x_1, 0)$

$u = -x_1 x_2^2 - kx_2$

$\dot{V} = -kx_2^2$

$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -kx_2 - x_1 x_2^2 \end{cases}$ (*)

$E = \{x \in \mathbb{R}^2,\ x_2 = 0\} \Rightarrow M = \{x_1 \in \mathbb{R},\ x_2 = 0\}$ (*)

choose another

$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4$

$\dot{V} = x_1\dot{x}_1 + x_2^3 \dot{x}_2$

$= x_1 x_2^3 + x_2^3 u.$

$u = -x_1 - kx_2$ (*)

$\Rightarrow \boxed{\dot{V} = -kx_2^4}$

$E = \{x \in \mathbb{R}^2,\ x_2 = 0\}$

$M = \{(0,0)\}$

Energy shaping (nonlinear spring)

$x_1 = x$, $x_2 = \dot{x}$

Friction-less system: $m\ddot{x} = -z(x) + u$

$z(x) = kx^3$ $\quad z(x) \in [0, \infty]$

if $z(x) = kx$

$m = 1$

Total energy : $E = \frac{1}{2}\dot{x}^2 + \int_0^x z(\sigma)\sigma$ → potential energy

Energy derivative along trajectories: $\dot{E} = \dot{x}u$ (*)

Control the energy to some desired level E_d

New Lyapunov function: $V = \frac{1}{2}(E - E_d)^2$ $\qquad \dot{V} = \dot{E}(E - E_d)$

p.d for $\dot{E} = \dot{x}u$. $\qquad = \dot{x}(E - E_d) \cdot u$

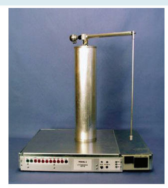
$u = -(E - E_d)$

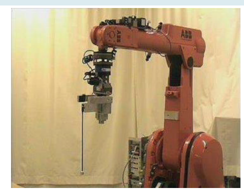
$\dot{V} = -\dot{x}(E - E_d)^2$

$u = -\dot{x}(E - E_d)$

$\dot{V} = -\dot{x}^2 (E - E_d)^2$

Energy shaping (swing-up control)





Rough outline of method to get the pendulum to the upright pcsition

- Find expression for total energy E of the pendulum (potential energy + kinetic energy)
- Let E_n be energy in upright position.
- Look at deviation $V = \frac{1}{2}(E - E_n)^2 \geq 0$
- Find "swing strategy" of control torque u such that $\dot{V} \leq 0$

Exact feedback linearization

- Find state feedback $u = u(x, v)$ so that the nonlinear affine in the control system
$$\dot{x} = f(x) + g(x)u$$
turns into the linear system
$$\boxed{\dot{x} = Ax + Bv}$$
and then apply linear control design method.
- Not all system can be exactly linearized. There are systems that their state needs to transformed to become linearizable:
  - first find $\boxed{z = T(x)}$ such that $\dot{z} = F(z) + G(z)u$
  - then find u that $\dot{z} = Az + Bv$
  - Design v as linear feedback controller with feedback of z.

Exact feedback linearization

- Find state feedback $u = u(x, v)$ so that the nonlinear affine in the control system
$$\dot{x} = f(x) + g(x)u$$
turns into the linear system
$$\dot{x} = Ax + Bv$$
and then apply linear control design method.
- Ax can be:
  - The linear part of the nonlinear system, e.g. $f(x) = Ax + \bar{f}(x)$
  - The linearized nonlinear system, e.g $f(x) = Ax + \bar{f}(x)$ where $\bar{f}(x) - \frac{\partial f}{\partial x}x$.
  - A desired linear dynamics specification.

Exact feedback linearization

- Relative degree 1: For $g(x)$ square and invertible

$$\dot x = f(x) + g(x)u \qquad x\in\mathbb{R}^n \quad u\in\mathbb{R}^n$$

$$u = g^{-1}(x)\left[-f(x)+v\right]$$

First order integrator

$$\dot x = v$$

- Relative degree n

$$\xi^{(n)} = f(\xi,\dot\xi,\cdots,\xi^{(n-1)}) + g((\xi,\dot\xi,\cdots,\xi^{(n-1)}))u$$

$$x = \left[\xi,\dot\xi,\cdots,\xi^{(n-1)}\right]^T$$

n-th order integrator

$$\xi^{(n)} = v$$

$$u = h(x_2) + z(x_1) + v$$

$$u = g^{-1}(x)\left[-f(x)+v\right]$$

$$\dot x = \begin{bmatrix} \mathbf{0}_{n-1} & \mathbf{I}_{n-1} \\ 0 & \mathbf{0}^T_{n-1} \end{bmatrix} x + \begin{bmatrix} \mathbf{0}_{n-1} \\ f(x) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n-1} \\ g(x) \end{bmatrix} u$$

$$\dot x = \begin{bmatrix} \mathbf{0}_{n-1} & \mathbf{I}_{n-1} \\ 0 & \mathbf{0}^T_{n-1} \end{bmatrix} x + \begin{bmatrix} \mathbf{0}_{n-1} \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x \qquad \dot x = \begin{bmatrix} 0 \\ v \end{bmatrix}$$

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -h(x_2) - z(x_1) + u \end{cases} \Rightarrow \left\{ \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -h(x_2) - z(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right.$$

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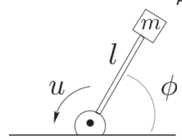
Exact feedback linearization



An inverted pendulum controlled by a motor torque u at the joint:

$$\ddot\phi(t) = \frac{g}{l}\sin(\phi(t)) + \frac{1}{ml^2}u,$$

$$x_1 = \phi, \quad x_2 = \dot\phi$$

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = \frac{g}{l}\sin(x_1) - \frac{1}{ml^2}u \end{cases}$$

Control structure for exact feedback linearization: $u = ml^2\left[\frac{g}{l}\sin(x_1) - v\right]$

$$\dot x_1 = x_2$$

$$\dot x_2 = \frac{g}{l}\cancel{\sin(x_1)} - \left[\cancel{\frac{g}{l}\sin(x_1)} - v\right] = v$$

Then v is chosen as $v = -k_1(\phi - \delta) - k_2\dot\phi$

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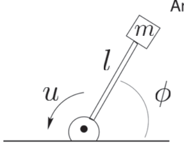
Exact feedback linearization and control-design based on linearization



An inverted pendulum controlled by a motor torque u at the joint:

$$\ddot\phi(t) = \frac{g}{l}\sin(\phi(t)) + \frac{1}{ml^2}u,$$

$$\dot x_1 = x_2$$

$$\dot x_2 = \frac{g}{l}\sin(x_1) + \frac{1}{ml^2}u$$

Control structure for exact feedback linearization: $u = ml^2\left[-\frac{g}{l}\sin(x_1) + v\right]$

Closed loop system:

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = v \end{cases} \quad \text{Linear}$$

Control design based on linearization: $u = ml^2\left[-\frac{g}{l}\sin\delta + v\right]$

δ is where we want to stabilize the system.

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = v + \frac{g}{l}(\sin(x_1) - \sin(\delta)) \end{cases} \quad \text{Non-linear system}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\phi & 0 \end{bmatrix} \quad \Rightarrow \quad \frac{\partial f}{\partial x}(\delta,0) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\delta & 0 \end{bmatrix}$$

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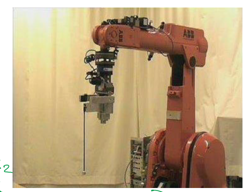
Multi–joint robot control with exact feedback linearization

Dynamic model of the robotic arm:

$$M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + G(\theta) = u, \qquad \theta\in R^n$$



Called *fully* actuated if n indep. actuators,

$M(\theta)$	$n\times n$ inertia matrix, $M = M^T > 0$
$C(\dot\theta,\theta)\dot\theta$	$n\times 1$ vector of centrifugal and Coriolis forces
$G(\theta)$	$n\times 1$ vector of gravitation terms

Design a controller so that $\theta \longrightarrow \theta_r$.

Inverse dynamics approach.

$$v = -k_p(x_1 - x_{1r}) - k_v x_2$$

$$= -K_p(\theta - \theta_r) - K_v\dot\theta \qquad K_p = \begin{bmatrix} k_{p1} & & \\ & \ddots & \\ & & k_{pn} \end{bmatrix} \quad K_v$$

$$M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + G(\theta) = u$$

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = M(x_1)^{-1}\left(-C(x_1,x_2)x_2 - G(x_1) + u\right) \end{cases}$$

$$u = M(x_1)\left(+C(x_1,x_2)x_2 + G(x_1) + v\right)$$

$$\dot x_2 = \cancel{M(x_1)^{-1}}\,\cancel{M(x_1)}\,v = v$$

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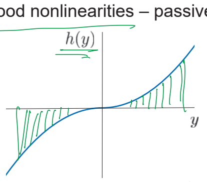
Should I cancel or not?

- Good nonlinearities – passive



$[0,\infty]$

$$\ddot x = -h(\dot x) - z(x) + u$$

$h(\dot x)$: nonlinear damping; $z(x)$: nonlinear spring

$$\dot x h(\dot x) \geq 0$$

Passive: energy absorbed by the damper is positive

$$\dot x z(x) = \frac{dP}{dt}$$

Passive: energy stored in the spring is positive

$$P = \int_0^x z(\sigma)\sigma$$

Total energy as Lyapunov function: $V = \frac{1}{2}\dot x^2 + \int_0^x z(\sigma)d\sigma$

Energy derivative along trajectories: $\dot V = -\dot x h(\dot x) + \dot x u$

$$u = -\varphi_D(\dot x) \qquad \varphi_D(\dot x)\in(0\quad\infty)$$

$$u = -kx$$

$$\dot V = -\dot x h(\dot x) - \dot x\varphi_D(\dot x)$$

$$V_1 = \frac{1}{2}\dot x^2 + \int_0^x z(\sigma)\,d\sigma + \int_0^x \phi_P(\sigma)\,d\sigma$$

What if I choose: $u = -\varphi_P(x)$

e.g. $u = -kx$, $\phi_P(x)\in[0,\infty)$

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Should I cancel or not?

Total energy as Lyapunov function: $V = \frac{1}{2}\dot x^2 + \int_0^x z(\sigma)\sigma$

Energy derivative along trajectories: $\dot V = -\dot x h(\dot x) + \dot x u$

$$u = -\varphi_D(\dot x) \qquad \varphi_D(\dot x)\in(0\quad\infty)$$

$$\dot V = -\dot x h(\dot x) - \dot x\varphi_D(\dot x)$$

LaSalle: V p.d., $\dot V \leq 0 \Longrightarrow M = \{x=0, \dot x = 0\}$ is maximum invariant set.

What if I choose: $u = -\varphi_P(x)$

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Robot manipulator – Example revisited with Lyapunov-based design

Dynamic model of the robotic arm:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = u, \qquad \theta \in R^n$$

Called *fully* actuated if n indep. actuators,

$M(\theta)$ — $n \times n$ inertia matrix, $M = M^T > 0$
$C(\dot{\theta},\theta)\dot{\theta}$ — $n \times 1$ vector of centrifugal and Coriolis forces
$G(\theta)$ — $n \times 1$ vector of gravitation terms

Design a controller so that $\theta \longrightarrow \theta_r$.

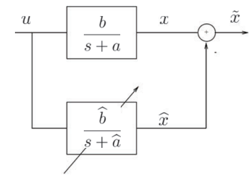
Inverse dynamics approach.

Another notable property:

$$S(\theta,\dot{\theta}) := \dot{M}(\theta) - 2C(\dot{\theta},\theta) = -S^T(\theta,\dot{\theta})$$

Adaptive noise cancellation

$\begin{cases} \dot{x} + ax &= bu \\ \dot{\widehat{x}} + \widehat{a}\widehat{x} &= \widehat{b}u \end{cases}$. Design adaptation law so that $\widetilde{x} := x - \widehat{x} \to 0$



Adaptation laws or update laws: $\dot{\widehat{a}} = ...$, $\dot{\widehat{b}} = ...$

Introduce $\widetilde{x} = x - \widehat{x}$, $\widetilde{a} = a - \widehat{a}$, $\widetilde{b} = b - \widehat{b}$.
What are the dynamics of the error?

Let us try the Lyapunov function $\begin{cases} V = \frac{1}{2}(\widetilde{x}^2 + \gamma_a \widetilde{a}^2 + \gamma_b \widetilde{b}^2) \\ \dot{V} = \end{cases}$

What do we prove if $\dot{V} \le 0$?

Are $\widetilde{a}$ and $\widetilde{b}$ proved to converge?

Simplified Adaptive control

$\begin{cases} \dot{x} &= \theta x^2 + u \\ u &= -\hat{\theta}(t)x^2 + v \end{cases}$

Set $\hat{\theta}(t) = \theta$
What principle of design is used?

Design:

- an update law for $\hat{\theta}$, $\dot{\hat{\theta}} = ...$
- a control signal $v(x)$

such that $x \to 0$

Introduce the new state $\tilde{\theta} = \theta - \hat{\theta}$.
Find $\dot{x} = f(x, \tilde{\theta}, v)$

Let us try the Lyapunov function $\begin{cases} V = \frac{1}{2}(x^2 + \gamma\tilde{\theta}^2) \\ \dot{V} = \end{cases}$

What do we prove if $\dot{V} \le 0$?

FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 10: Optimal control

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Simplified Adaptive control

$\begin{cases} \dot{x} &= \theta x^2 + u \\ u &= -\hat{\theta}(t)x^2 + v \end{cases}$ $\Rightarrow \dot{x} = \tilde{\theta}x^2 + v$

Set $\hat{\theta}(t) = \theta$
What principle of design is used?

$\rightarrow \dot{x} = \theta x^2 + u$
$u = -\theta x^2 + v$
$v = -kx$, $k > 0$
$\Rightarrow$ such that $x \to 0$ where θ is a constant.

$\dot{x} = -kx \Rightarrow x \to 0$ exponentially

Design:

- an update law for $\hat{\theta}$, $\dot{\hat{\theta}} = ...$
- a control signal $v(x)$ $= -kx$

such that $x \to 0$

Introduce the new state $\tilde{\theta} = \theta - \hat{\theta}$. $\dot{\tilde{\theta}} = \dot{\theta} - \dot{\hat{\theta}}$, $\dot{\theta} = 0$
Find $\dot{x} = f(x, \tilde{\theta}, v)$
$\boxed{\dot{\tilde{\theta}} = -w}$

$\begin{cases} \dot{x} = \tilde{\theta}x^2 + v \\ \dot{\tilde{\theta}} = -w \end{cases}$

Let us try the Lyapunov function $\begin{cases} V = \frac{1}{2}(x^2 + \gamma\tilde{\theta}^2) \\ \dot{V} = x\dot{x} + \gamma\tilde{\theta}\dot{\tilde{\theta}} \end{cases}$ $= x(\tilde{\theta}x^2 + v) - \gamma\tilde{\theta}w$
$= \tilde{\theta}x^3 - \gamma w\tilde{\theta} + xv$ $v = -kx$
$= -kx^2$

What do we prove if $\dot{V} \le 0$? $w = \frac{1}{\gamma}x^3$

$\dot{V} \le 0 \rightarrow$ Lasalle $M = \{\tilde{\theta} \in R, x = 0\}$

Outline

- Static optimization
- Problem formulation
- Maximum principle
- Examples

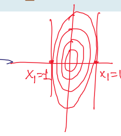
Optimal Control

Idea: Formulate the design problem as optimization problem

- + Gives systematic design procedure
- + Can use on nonlinear models
- + Can capture limitations etc as constraints
- – Hard to find suitable criterium?!
- – Can be hard to find the optimal controller

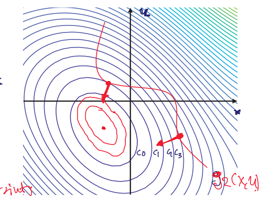
Recap static optimization

Optimization under constraints:



$$\begin{array}{ll}\min_{x,u} & g_1(x,u)\\ \text{s.t.} & g_2(x,u)=0\end{array}$$

$g_1(x,u)=c$



Necessary conditions for optimality:

- ∇g_1 points in the same direction as ∇g_2
- $g_2(x,u)=0$

λ Lagrange multipliers

$\#\lambda = \#$ constraints

Lagrangian: $\mathcal{L}(x,u,\lambda)=\underbrace{g_1(x,u)}_{\text{cost}}+\lambda^T \underbrace{g_2(x,u)}_{\text{constraints}}$

Lagrange multipliers

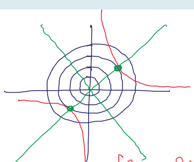
- $\frac{\partial \mathcal{L}}{\partial x}=0,\ \frac{\partial \mathcal{L}}{\partial u}=0,\ \frac{\partial \mathcal{L}}{\partial \lambda}=0$
- $\frac{\partial^2 \mathcal{L}}{\partial x^2}>0,\ \frac{\partial^2 \mathcal{L}}{\partial u^2}>0$

$$\begin{bmatrix}\frac{\partial \mathcal{L}}{\partial x}\\ \frac{\partial \mathcal{L}}{\partial u}\\ \frac{\partial \mathcal{L}}{\partial \lambda}\end{bmatrix}=0 \qquad \begin{bmatrix}\frac{\partial g_1(x,u)}{\partial x}+\lambda\frac{\partial g_2(x,u)}{\partial x}\\ \frac{\partial g_1(x,u)}{\partial u}+\lambda\frac{\partial g_2(x,u)}{\partial u}\\ g_2(x,u)=0\end{bmatrix}=0$$

Static optimization

$$\begin{array}{ll}\min_{x_1,x_2} & (x_1^2+x_2^2)\\ \text{s.t.} & \boxed{x_1x_2-1=0}\end{array} \to 1 \text{ constraint}$$

$\mathcal{L}=x_1^2+x_2^2+\lambda(x_1x_2-1)$



$$\begin{bmatrix}\frac{\partial g_1(x_1,x_2)}{\partial x_1}+\lambda\frac{\partial g_2(x_1,x_2)}{\partial x_1}\\ \frac{\partial g_1(x_1,x_2)}{\partial x_2}+\lambda\frac{\partial g_2(x_1,x_2)}{\partial x_2}\\ g_2(x_1,x_2)=0\end{bmatrix}=0 \Rightarrow \begin{cases}2x_1+\lambda x_2=0\\ 2x_2+\lambda x_1=0\\ x_1x_2-1=0\end{cases} \Rightarrow \begin{cases}(x_1-x_2)(x_1+x_2)=0 & (1)\\ x_1x_2-1=0 & (2)\\ 2x_1+\lambda x_2=0\end{cases}$$

$\begin{cases}2x_1=-\lambda x_2\\ 2x_2=-\lambda x_1\end{cases} \Rightarrow \frac{x_1}{x_2}=+\frac{x_2}{x_1} \Rightarrow x_1^2=x_2^2$

(1) 2 solutions

$x_1=x_2 \Rightarrow x_2^2=1 \Rightarrow x_2=\pm 1=x_1$

$x_1=-x_2 \Rightarrow x_2^2+1=0$ no real solutions

Maximum principle – no final time constraint

Optimization Problem (OP1)

Minimize $\int_0^{t_f}\overbrace{L(x(t),u(t))}^{\text{Trajectory cost}}\,dt+\overbrace{\phi(x(t_f))}^{\text{Final cost}}$ ← cost

where

$x(t)\in R^n,\quad u(t)\in U\subseteq R^m$ → inequality constraint

$\dot x(t)=f(x(t),u(t)),\qquad x(0)=x_0$

$0\le t\le t_f,\qquad$ given fixed end-time t_f

Lagrange multipliers

Theorem 18.2 of Glad/Ljung Assume that the (OP1) has a solution $\{u^*(t),x^*(t)\}$. Then

$$\min_{u\in U} H(x^*(t),u,\lambda(t))=H(x^*(t),u^*(t),\lambda(t)),\quad 0\le t\le t_f,$$

$H(x,u,\lambda)=L(x,u)+\lambda^T(t)f(x,u)$

where $\lambda(t)$ solves the **adjoint equation**

$$H_x=\frac{\partial H}{\partial x}=\left(\frac{\partial H}{\partial x_1}\quad\frac{\partial H}{\partial x_2}\ \dots\right)$$

$$\frac{d\lambda}{dt}=-H_x^T(x^*(t),u^*(t),\lambda(t)),\quad\text{with}\quad\lambda(t_f)=\phi_x^T(x^*(t_f))$$

Sketchy proof (Hamiltonian)

Minimize $\int_{t_0}^{t_f}\overbrace{L(x(t),u(t))}^{\text{Trajectory cost}}\,dt+\overbrace{\phi(x(t_f))}^{\text{Final cost}}$

where

$x(t)\in R^n,\quad u(t)\in U\subseteq R^m$

$\dot x(t)=f(x(t),u(t)),\qquad x(t_0)=x_0$

$t_0\le t\le t_f,\qquad$ given fixed end-time t_f

$f-\dot x=0$

Optimal Control Problem

$$\min_u J=\min_u\left\{\phi(x(t_f))+\int_{t_0}^{t_f}L(x,u)\,dt\right\}$$

subject to $\dot x=f(x,u),\quad x(t_0)=x_0$

Functions of time the constraint is satisfied over the assumed period of time

$$J=\phi(x(t_f))+\int_{t_0}^{t_f}\left(\boxed{L(x,u)+\lambda^T(f}-\dot x)\right)dt$$

$$=\phi(x(t_f))-\left[\lambda^Tx\right]_{t_0}^{t_f}+\int_{t_0}^{t_f}\left(H+\dot\lambda^Tx\right)dt$$

$H(x,u,\lambda)=L(x,u)+\lambda^Tf(x,u)$

Sketchy proof (Calculus of variation)

Variation of J:

$$\delta J=\left[\left(\frac{\partial\phi}{\partial x}-\lambda^T\right)\delta x\right]_{t=t_f}+\int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x}+\dot\lambda^T\right)\delta x+\frac{\partial H}{\partial u}\delta u\right]dt$$

Necessary conditions for local minimum ($\delta J=0$)

$$\boxed{\lambda(t_f)^T}=\boxed{\frac{\partial\phi}{\partial x}}\Big|_{t=t_f}\qquad \dot\lambda^T=-\underbrace{\frac{\partial H}{\partial x}}_{H_x}\qquad \frac{\partial H}{\partial u}=0$$

- λ specified at $t=t_f$ and x at $t=t_0$
- Two Point Boundary Value Problem (TPBV)
- For sufficiency $\frac{\partial^2 H}{\partial u^2}\ge 0$

Summary of the approach

Performance, cost function $J(x_0) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x,u)\,dt$

System dynamics $\dot{x} = f(x,u),\ x(t_0) = x_0$

No final-time constraint but final time is a free variable

Hamiltonian $H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$

State equation $\dot{x} = f(x,u), \quad x(t_0) = x_0$ **(1)**

Co-state, adjoint equation $\dot{\lambda} = H_x^T(x,u,\lambda), \quad \lambda(t_f) = \phi_x^T[x(t_f)]$ **(2)**

$\phi_x = \left[\frac{\partial \phi}{\partial x_1} \cdots \frac{\partial \phi}{\partial x_n}\right]$

Hamiltonian minimization with respect to u $\min_{u \in U} H(u)$ **(3)**

We can often first eliminate the control input u(t) by (3)

Remarks

- The Maximum Principle gives **necessary** conditions
- A pair $(u^*(\cdot), x^*(\cdot))$ is called **extremal** the conditions of the Maximum Principle are satisfied.
- Many extremals can exist.
- The maximum principle gives all possible candidates.
- However, there might not exist a minimum!

Example

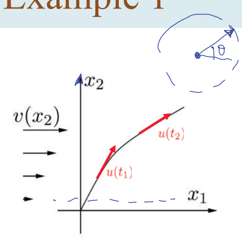
Minimize $x(1)$ when $\dot{x}(t) = u(t)$, $x(0) = 0$ and $u(t)$ is free

Why doesn't there exist a minimum?

Example 1

$\min \int_0^{t_f} \mathcal{L}(x,u) + \phi(x(t_f))$

$\phi(x(T)) = -x_1(T)$



$$\min_{u:[0,t_f)\to U} \ -x_1(T)$$
$$\dot{x}_1 = v(x_2) + u_1$$
$$\dot{x}_2 = u_2$$
$$x_1(0) = 0$$
$$x_2(0) = 0$$
$$\boxed{u_1^2 + u_2^2 = 1}$$

- Speed of water $v(x_2)$ in x_1 direction with $\frac{\partial v(x_2)}{\partial x_2} = 1$ $= x_2$
- Move (sail) maximum distance in x_1-direction in fixed time T
- Rudder angle control: $u \in U := \{(u_1, u_2): \quad u_1^2 + u_2^2 = 1\}$

Example 1

Hamiltonian:

$$H = 0 + \lambda^T f = [\lambda_1(t)\ \lambda_2(t)] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \lambda_1(v(x_2) + u_1) + \lambda_2 u_2$$

Adjoint equation:

H is not a function of x_1.

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -\partial H/\partial x_1 \\ -\partial H/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -v'(x_2)\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

$v'(x_2) = 1$

with boundary conditions

$$\begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \end{bmatrix} = \begin{bmatrix} \partial\phi/\partial x_1|_{x=x^*(t_f)} \\ \partial\phi/\partial x_2|_{x=x^*(t_f)} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$\phi(x_1) = -x_1$

$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1(t) = -1, \forall t$

$\dot{\lambda}_2 = -\lambda_1 \Rightarrow \dot{\lambda}_2 = 1$

$\lambda_1(T) = -1$

$\lambda_2(T) = 0$

$\dot{\lambda}_2 = 1 \Rightarrow \lambda_2 = t + C$

$\lambda_2 = t$

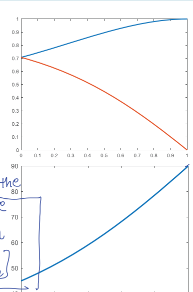
$\lambda_2 = t - T$

Example 1

$\min a^T b$, $\|b\| = 1$ $\sim$ when $\theta = \pi$ $\longleftrightarrow$ $a^T b|_{\theta=\pi} = -\|a\|$

$a^T b = \|a\| \cos\theta$

Solution of the co-state $\lambda_1(t) = -1, \quad \lambda_2(t) = t - T.$



Optimality: Control signal should solve

$$\min_{u_1^2+u_2^2=1} \lambda_1(v(x_2) + u_1) + \lambda_2 u_2$$

Minimize $\lambda_1 u_1 + \lambda_2 u_2$ so that (u_1, u_2) has length 1

$\min_{u_1^2+u_2^2=1} [\lambda_1, \lambda_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ → unit

$$u_1(t) = -\frac{\lambda_1(t)}{\sqrt{\lambda_1^2(t) + \lambda_2^2(t)}}, \quad u_2(t) = -\frac{\lambda_2(t)}{\sqrt{\lambda_1^2(t) + \lambda_2^2(t)}}$$

when u is in the opposite direction with $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$$u_1(t) = \frac{1}{\sqrt{1 + (t-T)^2}}, \quad u_2(t) = \frac{T - t}{\sqrt{1 + (t-T)^2}}$$

See fig 18.1 for plots

Remark: It can be shown that this optimal control problem has a minimum. Hence it must be the one we found, since this was the only solution to MP

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Lecture 11: Optimal control 2

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Example 2

$$\min \int_0^1 u^4 dt + x(1)$$
$$\dot{x} = -x + u, \quad x(0) = 0$$

$$\min \int_0^{t_f} \overbrace{L(x(t), u(t))}^{\text{Trajectory cost}} \, dt + \overbrace{\phi(x(t_f))}^{\text{Final cost}}$$

where

$x(t) \in R^n$

$u(t) \in U \subseteq R^m$

$\dot{x}(t) = f(x(t), u(t))$

$x(0) = 0$

Hamiltonian:

$$H = L + \lambda^T \cdot f = u^4 + \lambda(-x + u)$$

Adjoint equation:

Final value problem

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -(-\lambda) \quad \Longrightarrow \quad \lambda(t) = Ce^t$$

$$\lambda(t_f) = \frac{\partial \phi}{\partial x} = 1 \quad \lambda(1) = 1 = Ce^1 \quad \Longrightarrow \quad \lambda(t) = e^{t-1}$$

$\Rightarrow C = e^{-1}$

Optimality: $H_u = .. \frac{\partial H}{\partial u} = 4u^3 + \lambda$

$H_u = 0 \Rightarrow u = \sqrt[3]{-\frac{\lambda}{4}}$

Maximum principle

Optimization Problem (OP2)

Minimize $\int_0^{t_f} \overbrace{L(x(t), u(t))}^{\text{Trajectory cost}} \, dt + \overbrace{\phi(t_f, x(t_f))}^{\text{Final cost}}$

where

$x(t) \in R^n, \quad u(t) \in U \subseteq R^m, \quad 0 \le t \le t_f$

$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$

$\psi(t_f, x(t_f)) = 0$

Assume that OP2 has a solution. Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$ and a vector $\mu \in R^r$ such that $[n_0 \;\; \mu^T] \neq 0$ and

$$H(x, u, \lambda) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

$$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation**
$$\begin{cases} \dot{\lambda}(t) &= -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\ \lambda(t_f) &= n_0 \phi_x^T(t_f, x^*(t_f)) + \psi_x^T(t_f, x^*(t_f))\mu \end{cases}$$

If the end time t_f is given, then $H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$.

If the end time t_f is free:

$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = -n_0 \phi_t(t_f, x^*(t_f)) - \mu^T \psi_t(t_f, x^*(t_f))$.

Remarks

- Can scale $n_0, \mu, \lambda(t)$ by the same constant

$$\begin{cases} \dot{\lambda}(t) &= -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\ \lambda(t_f) &= n_0 \phi_x^T(t_f, x^*(t_f)) + \psi_x^T(t_f, x^*(t_f))\mu \end{cases}$$

$$H(x, u, \lambda) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

- Can reduce to two cases $\quad H(x, u, \lambda) = n_0 L(x, u) + \lambda^T(t) f(x, u)$
  - $n_0 = 1$ (normal) $\quad$ If $n_0 > 0$, renormalize to $n_0 = 1$. Only existence of positive n_0 matters.
  - $n_0 = 0$ (abnormal, since L and ϕ don't matter)

- Fixed time t_f and no end constraints $\Rightarrow$ normal case

Optimal control (Linear Control Systems with quadratic running cost)

Performance, cost function $\quad J(u) = \frac{1}{2} x^T(t_f) P(t_f) x(t_f) + \frac{1}{2} \int_0^{t_f} x^T Q x + u^T R u \, dt$

System dynamics (dynamic constraint) $\quad \dot{x} = f(x, u), \; x(t_0) = x_0$

Hamiltonian $\quad H(x, u, \lambda) = \frac{1}{2}(x^T Q x + u^T R u) + \lambda^T(Ax + Bu)$

Hamiltonian minimization with respect to u $\quad \frac{\partial H(u)}{\partial u} = u^T R + \lambda^T B \to u = -R^{-1} B^T \lambda \quad \lambda(t) = P(t)x(t)$

State equation
Co-state, adjoint equation
$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \begin{matrix} x(t_0) = x_0 \\ \lambda(t_f) = P(t_f)x(t_f) \end{matrix}$$

$$\dot{P}(t) = P(t)A + A^T P(t) + P(t) B R^{-1} B^T(t) P(t) - Q$$

$P(t_f)$ known by the cost function

Example – optimal heating (minimum fuel problem)

$$\min \int_0^{t_f=1} P(t) \, dt$$

s.t. $\quad \dot{T} = P - T, \quad T(0) = 0$

$\quad 0 \le P \le P_{max}$

$\quad T(1) = 1$

T temperature

P heat effect

Hamiltonian

$$H = n_0 P + \lambda(P - T)$$

Adjoint equation

$$\dot{\lambda}^T = -H_T = -\frac{\partial H}{\partial T} = \lambda^T \qquad \lambda(1) = \mu$$

$$\Rightarrow \lambda(t) = \mu e^{t-1}$$
$$\Rightarrow H = \underbrace{(n_0 + \mu e^{t-1})}_{\sigma(t)} P - \lambda T$$

H cannot be minimized by $H_u = 0$

The constraint on P makes the problem well-defined

At optimality

$$P^*(t) = \begin{cases} 0, & \sigma(t) > 0 \\ P_{max}, & \sigma(t) < 0 \end{cases}$$

Example – optimal heating

$$\min \int_0^{t_f=1} P(t) \, dt$$

s.t. $\quad \dot{T} = P - T, \quad T(0) = 0$

$\quad 0 \le P \le P_{max}$

$\quad T(1) = 1$

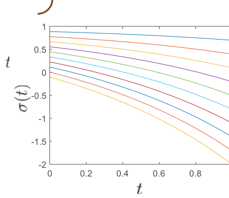
$$H = \sigma(t) P - \lambda T$$
$$\sigma(t) = n_0 + \underbrace{\mu e^{t-1}}_{\lambda}$$

$$P^*(t) = \begin{cases} 0, & \sigma(t) > 0 \\ P_{\max}, & \sigma(t) < 0 \end{cases}$$

$$\left.\begin{matrix} \mu > 0 \Rightarrow \sigma(t) > 0 \text{ for all } t \\ \mu = 0 \Rightarrow \lambda(t) = 0 \;\forall t \Rightarrow \sigma(t) = n_0 > 0 \;\forall t \end{matrix}\right\} \begin{matrix} \Rightarrow P \equiv 0 \Rightarrow \dot{T} = -T, \, T(0) = 0 \\ \Rightarrow T(1) = 0 \neq 1 \end{matrix}$$

$\mu < 0 \Rightarrow \sigma(t)$ strictly decreasing for all t

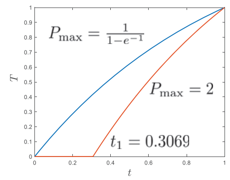
$$\Rightarrow \dot{T} = -(T - P_{\max}), \; T(t_1) = 0$$
$$\Rightarrow T(t) = \left[1 - e^{-(t - t_1)}\right] P_{\max}$$

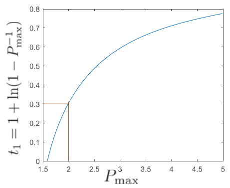


T temperature

P heat effect

Example – optimal heating





If $t_1 = 0$ (no switching)

$$P_{\max} = \frac{1}{1-e^{-1}}$$

If $0 < t_1 < 1$ (switching from 0 to $P_{\max}$)

$$T(1) = \left[1 - e^{-(1-t_1)}\right] P_{\max} = 1$$

$$t_1 = 1 + \ln(1 - P_{\max}^{-1})$$

$$P_{\max} > \frac{1}{1-e^{-1}}$$

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Lecture 12: From optimal control to nonlinear control

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Outline

Linear Quadratic Control
Time Optimal Control
Model Predictive Control

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Maximum principle – no final time constraint

Optimization Problem (OP2)

Minimize $\int_0^{t_f} \overbrace{L(x(t),u(t))}^{\text{Trajectory cost}}\, dt + \overbrace{\phi(t_f, x(t_f))}^{\text{Final cost}}$

where

$x(t) \in R^n, \quad u(t) \in U \subseteq R^m, \quad 0 \le t \le t_f$

$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0$

$\psi(t_f, x(t_f)) = 0$

Assume that OP2 has a solution. Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$ and a vector $\mu \in R^r$ such that $[n_0 \ \ \mu^T] \ne 0$ and

$$H(x,u,\lambda) = n_0 L(x,u) + \lambda^T(t) f(x,u)$$

$$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation** $\begin{cases} \dot{\lambda}(t) & = -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\ \lambda(t_f) & = n_0 \phi_x^T(t_f, x^*(t_f)) + \psi_x^T(t_f, x^*(t_f))\mu \end{cases}$

If the end time t_f is free, then $H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$.
If the end time t_f is given:
$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = -n_0 \phi_t(t_f, x^*(t_f)) - \mu^T \psi_t(t_f, x^*(t_f))$.

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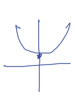
Optimal control (Linear Control Systems with quadratic running cost)

Performance, cost function
Free-final-state but in the perfromance function
$$J(u) = \frac{1}{2} x^T(t_f) P(t_f) x(t_f) + \frac{1}{2}\int_0^{t_f} x^T Q x + u^T R u \, dt$$

System dynamics (dynamic constraint)
$$\dot{x} = f(x,u), \; x(t_0) = x_0$$

Hamiltonian
$$H(x,u,\lambda) = \tfrac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu) = u^T A_1 u + b^T u + e$$

Hamiltonian minimization with respect to u
$$\frac{\partial H(u)}{\partial u} = u^T R + \lambda^T B \to \boxed{u = -R^{-1} B^T \cancel{P}} \times$$



State equation
Co-state, adjoint equation
$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \begin{matrix} x(t_0) = x_0 \\ \lambda(t_f) = P(t_f) x(t_f) \end{matrix}$$

$\lambda(t) = P(t) x(t)$

$$\begin{cases} \dot{x}(t) = (A(t) - BR^{-1}B^T P(t)) x(t), \quad x(0) \text{ given} \\ \dot{P}(t) = P(t)A + A^T P(t) + P(t) B R^{-1} B^T(t) P(t) - Q \end{cases}$$

$P(t_f)$ known by the cost function

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Optimal control (Linear Control Systems with quadratic running cost and fixed final state)

Performance, cost function
$$J(u) = \frac{1}{2}\int_0^{t_f} u^T R u \, dt$$

Constraint on
$$\psi(x(t_f)) = 0 \to x(t_f) = r$$

System dynamics (dynamic constraint)
$$\dot{x} = f(x,u), \; x(t_0) = x_0$$

Hamiltonian
$$H(x,u,\lambda) = \tfrac{1}{2} u^T R u + \lambda^T (Ax + Bu)$$

Hamiltonian minimization with respect to u
$$\frac{\partial H(u)}{\partial u} = u^T R + \lambda^T B \to u = -R^{-1} B^T \lambda$$

$\dot{x} = -ax + f$

State equation
Co-state, adjoint equation
$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ O & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \begin{matrix} x(t_0) = x_0 \\ \lambda(t_f) \text{ ??} \end{matrix}$$

Steps: 1. solve backwards the second differential equation $\lambda(t) = e^{A^T(t_f - t)} \lambda(t_f)$
2. substitute the solution in the system dynamics and solve the initial value problem
3. use the constraint to find $\lambda(t_f)$
$x(t_f) = r$

$u^*(t)$

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Example – moving to the origin in minimum time

$$\min \int_0^{t_f} dt$$

s.t. $\ddot{x} = u, \quad x(0) = x_0,\ \dot{x}(0) = v_0$

$u \in [-1, 1]$

$x(t_f) = 0, \quad \dot{x}(t_f) = 0$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

$\min t_f$

Hamiltonian: $H = n_0 + \lambda_1 x_2 + \lambda_2 u$

Adjoint equation:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -H_x^T = \begin{bmatrix} -\frac{\partial H}{\partial x_1} \\ -\frac{\partial H}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix} \qquad \begin{cases} \lambda_1(t_f) = \mu_1 \\ \lambda_2(t_f) = \mu_2 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1(t) = \mu_1 \\ \lambda_2(t) = \mu_1(t - t_f) + \mu_2 \end{cases}$$

At optimality: $\min_u H(u) \equiv \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\sigma(t) = \lambda_2(t)} u + n_0 + \lambda_1 x_2$

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$

Conditions: $H(t_f) = 0 \Rightarrow \mu_2 u^*(t_f) = -n_0$

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Example – minimum time control

$$\min \int_0^{t_f} dt$$

s.t. $\ddot{x} = u, \quad x(0) = x_0,\ \dot{x}(0) = v_0$

$u \in [-1, 1]$

$x(t_f) = 0, \quad \dot{x}(t_f) = 0$

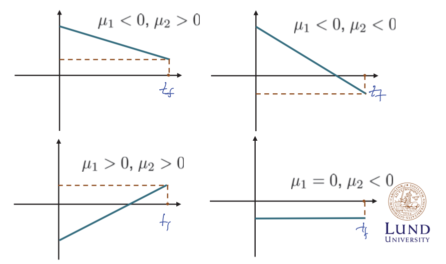
$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

$H = (\mu_1(t - t_f) + \mu_2)u + n_0 + \lambda_1 x_2$

$\mu_2 u^*(t_f) = -n_0$

$\sigma(t) = \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\lambda_2(t)}$

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$



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Example – minimum time control

$$\min \int_0^{t_f} dt$$

s.t. $\ddot{x} = u, \quad x(0) = x_0,\ \dot{x}(0) = v_0$

$u \in [-1, 1]$

$x(t_f) = 0, \quad \dot{x}(t_f) = 0$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

$\Rightarrow \frac{dx_1}{dx_2} = \frac{x_2}{u} \Rightarrow u\,dx_1 = x_2\,dx_2$

$\Rightarrow \begin{cases} x_1 = -\frac{x_2^2}{2} + c, & \sigma > 0 \ (u = -1) \\ x_1 = \frac{x_2^2}{2} + c, & \sigma < 0 \ (u = +1) \end{cases}$

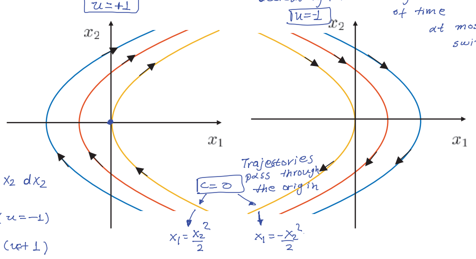
$H = (\mu_1(t - t_f) + \mu_2)u + n_0 + \lambda_1 x_2$

$\mu_2 u^*(t_f) = -n_0$

$\sigma(t) = \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\lambda_2(t)}$ → monotonically decreasing / increasing function of time

at most one switching

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$



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Example – minimum time control

See Ex. 7.11

$$\min \int_0^{t_f} dt$$

s.t. $\ddot{x} = u, \quad x(0) = x_0,\ \dot{x}(0) = v_0$

$u \in [-1, 1]$

$x(t_f) = 0, \quad \dot{x}(t_f) = 0$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

$H = (\mu_1(t - t_f) + \mu_2)u + n_0 + \lambda_1 x_2$

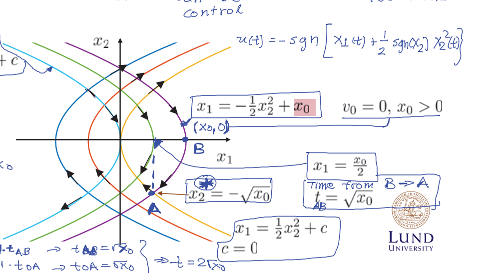
$\mu_2 u^*(t_f) = -n_0$

$\sigma(t) = \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\lambda_2(t)}$

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$

Can be seen as feedback control

$u(t) = -\mathrm{sgn}\left[x_1(t) + \frac{1}{2}\mathrm{sgn}(x_2)\, x_2^2(t)\right]$



(*) Intersection of $\begin{cases} x_1 = -\frac{1}{2}x_2^2 + x_0 \\ x_1 = \frac{1}{2}x_2^2 \end{cases}$

$A\left(\frac{x_0}{2}, -\sqrt{x_0}\right)$

Time B → A, $u = -1$, $\dot{x}_2 = u$

$(v_A - v_B) = -1 \cdot t_{AB} \rightarrow t_{AB} = \sqrt{x_0}$

Time A → 0

$(v_0 - v_A) = 1 \cdot t_{0A} \rightarrow t_{0A} = \sqrt{x_0}$

$\Rightarrow t = 2\sqrt{x_0}$

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- Optimal control provides with trajectories that can be then used as references to a controller
- We mainly address input constraints, but there might be state constraints e.g. obstacles that need to be avoided
- A popular approach is Model Predictive Control

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From Continuous to Discrete Time

$\dot{x} = Ax + Bu$

$y = Cx$

Sampling time T

$A_d = e^{AT}$

$B_d = (A_d - I)A^{-1}B$

Euler Approximation

$A_d = I + TA$

$B_d = TB$

$x(t+1) = A_d x(t) + B_d u(t)$

$y(t) = Cx(t)$

$$J = \frac{1}{2}x^T(t_f)P(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x^TQx + u^TRu\,dt$$

The integral becomes sum

$$J = x^T(M)P(M)x(M) + \sum_1^{M-1} x^T(i)Qx(i) + u^T(i)Ru(i)$$

$$J = y^T(M)P(M)y(M) + \sum_1^{M-1} y^T(i)Qy(i) + u^T(i)Ru(i)$$

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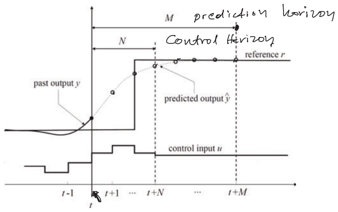
Receding horizon control – basic idea



1. At time instant t predict the response of the system over a prediction horizon M using inputs over a control horizon N.
2. Optimize a specified objective or cost function with respect to a control sequence $u(t+j), \quad j = 0, 1, \ldots, N-1$.
3. Apply the first control $u(t)$ and start over from 1 at next sample.

Unroll the cost

Minimize a cost function, V, of inputs and predicted outputs.

$$V = V(U_t, Y_t), \qquad U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \qquad Y_t = \begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix}$$

V often quadratic

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t \tag{1}$$

$\Longrightarrow$ linear controller

$$u(t) = -L\widehat{x}(t|t)$$

Optimization problem

$$J(t) = y^T(t+M)P(t+M)y(t+M) + \sum_{t+1}^{t+M-1} y^T(i)Qy(i) + u^T(i)Ru(i)\,dt$$

Minimize a cost function, V, of inputs and predicted outputs.

$$V = V(U_t, Y_t), \qquad U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \qquad Y_t = \begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix}$$

V often quadratic

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t$$

$\Longrightarrow$ linear controller

$$u(t) = -L\widehat{x}(t|t)$$

Prediction

Discrete-time model

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_v v_1(t) \\ y(t) &= Cx(t) + v_2(t) \end{aligned} \qquad t = 0, 1, \ldots$$

Predictor (v unknown)

$$\begin{aligned} \widehat{x}(t+k+1|t) &= A\widehat{x}(t+k|t) + Bu(t+k) \\ \widehat{y}(t+k|t) &= C\widehat{x}(t+k|t) \end{aligned}$$

Receding horizon control

- $\widehat{x}(t|t)$ is predicted by a standard Kalman filter, using outputs up to time t, and inputs up to time $t-1$.
- Future predicted outputs are given by

$$\begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix} = \begin{bmatrix} CA^M \\ \vdots \\ CA \end{bmatrix} \widehat{x}(t|t) + \begin{bmatrix} CB & CAB & CA^2B & \ldots \\ 0 & CB & CAB & \ldots \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} u(t+M-1) \\ \vdots \\ u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}$$

$$Y_t = D_x\widehat{x}(t|t) + D_u U_t$$

Receding horizon control

+ Flexible method
  - ⋆ Many types of models for prediction:
    - ⋆ state space, input–output, step response, nonlinear models
  - ⋆ MIMO
  - ⋆ Time delays
+ Can include constraints on input signal and states
+ Can include future reference and disturbance information
– On-line optimization needed
– Stability (and performance) analysis can be complicated

Receding horizon control

Limitations on control signals, states and outputs,

$$|u(t)| \le C_u \quad |x_i(t)| \le C_{x_i} \quad |y(t)| \le C_y,$$

leads to linear programming or quadratic optimization.
Efficient optimization software exists.

Design of MPC

- Model
- M (look on settling time)
- N as long as computational time allows
- If $N < M-1$ assumption on $u(t+N), \ldots, u(t+M-1)$ needed (e.g., $=0$, $=u(t+N-1)$.)
- Q_y, Q_u (trade-offs between control effort and performance)
- C_y, C_u constraints often given
- Sampling time

FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 13: Control Systems with discontinuities

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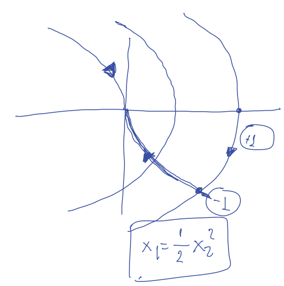
Outline

- ~~Time optimal control~~
- Sliding mode control
- Friction



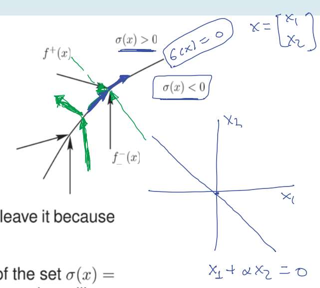
Sliding set

- **Sliding set:**

$$\dot{x} = \begin{cases} f^+(x) & \sigma(x) > 0 \\ f^-(x) & \sigma(x) < 0 \end{cases} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\sigma(x) = 0$

$$\left.\begin{aligned} \frac{\partial \sigma}{\partial x} f^+ < 0 \\ \frac{\partial \sigma}{\partial x} f^- > 0 \end{aligned}\right\} \quad f^+ \text{ and } f^- \text{ point towards } \sigma(x) = 0$$



- Once the trajectory hits the switching surface S, it cannot leave it because the vector fields on both sides point towards the surface
- If f^+ and f^- point "in the same direction" on both sides of the set $\sigma(x) = 0$, then the solution curves will just pass through and this region will not belong to the sliding set.
- If $f^+(x) = f^-(x)$ on $\sigma(x)$ then the sliding set is actually a set of equilibrium points

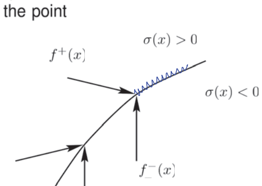
Sliding set

- The general behavior for the solution is to slide on $\sigma(x)$ (sliding mode.)
- The sliding motion can be described by noting that there is a unique convex combination of $f^+(x)$ and $f^-(x)$ that is tangent to $\sigma(x)$ at the point x.
- **Sliding dynamics:**

$$\dot{x} = \alpha(x) f^+(x) + (1 - \alpha(x)) f^-(x),$$

where $\alpha(x)$ is obtained from

$$\boxed{0 = \frac{d\sigma}{dt}} = \frac{\partial \sigma}{\partial x} \cdot \dot{x}$$
$$= \frac{\partial \sigma}{\partial x} f^-(x) + \alpha(x) \frac{\partial \sigma}{\partial x} \left[f^+(x) - f^-(x) \right].$$



- The fast switches will give rise to average dynamics that slide along the set where $\sigma(x) = 0$.

Example

$$\dot{x} = \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}x + \begin{bmatrix}1\\ 1\end{bmatrix}u = Ax + Bu$$

$$u = -\text{sgn}\sigma(x) = -\text{sgn}x_2 = -\text{sgn}(Cx)$$

which means that

$$\boxed{\dot{x} = \begin{cases} Ax - B, & x_2 > 0\\ Ax + B, & x_2 < 0\end{cases}}$$

Determine the *sliding set* and the *sliding dynamics*.

Example: Sliding set

$\frac{\partial\sigma}{\partial x} = [0\ 1]$

$\sigma(x)$

$$\dot{x}_1 = -x_2 + u = -x_2 - \text{sgn}(x_2)$$
$$\dot{x}_2 = x_1 - x_2 + u = x_1 - x_2 - \text{sgn}(x_2)$$

$\sigma(x) = 0$ $\quad f^+ = \begin{bmatrix}-x_2 - 1\\ x_1 - x_2 - 1\end{bmatrix}$ $\quad f^- = \begin{bmatrix}-x_2 + 1\\ x_1 - x_2 + 1\end{bmatrix}$

$\sigma(x) = x_2$

$\frac{\partial\sigma}{\partial x_1} = 0$

$\frac{\partial\sigma}{\partial x_2} = 1$

$\sigma(x) = 0$ $\to [0\ 1]$	$x_2 = 0$
$\frac{\partial\sigma}{\partial x}f^+ < 0$	$x_1 - x_2 - 1 < 0$ $\quad x_1 < 1$
$\frac{\partial\sigma}{\partial x}f^- > 0$	$x_1 - x_2 + 1 > 0$ $\quad x_1 > -1$

We thus have the sliding set $\{-1 < x_1 < 1,\ x_2 = 0\}$

Example: Sliding dynamics

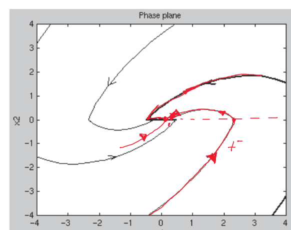
The *sliding dynamics* are the given by

$$\dot{x} = \alpha f^+ + (1-\alpha)f^- \quad \text{sliding dynamics}$$
$$0 = \frac{\partial\sigma}{\partial x}f^-(x) + \alpha(x)\frac{\partial\sigma}{\partial x}\left[f^+(x) - f^-(x)\right]$$

On the sliding set $\{-1 < x < 1,\ x_2 = 0\}$, this gives

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \alpha\begin{bmatrix}-x_2 - 1\\ x_1 - x_2 - 1\end{bmatrix} + (1-\alpha)\begin{bmatrix}-x_2 + 1\\ x_1 - x_2 + 1\end{bmatrix}$$
$$0 = x_1 - x_2 + 1 - 2\alpha$$

Eliminating α gives $\begin{cases}\boxed{\dot{x}_1 = -x_1}\\ \dot{x}_2 = 0\end{cases}$ Hence, any initial condition the sliding set will give exponential convergence to $x_1 = x_2 = 0$.



The dynamics along the sliding set in $\sigma(x) = 0$ can also be obtained by finding $u = u_{\text{eq}} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.
u_{eq} is called the **equivalent control**.

Sliding mode control

- Define an output $y = \sigma(x)$ such that the relative degree of the input-output relationship is 1 and the origin for the the system $\sigma(x) = 0$ is stable.

Pendulum: $\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -\theta\cos(x_1) + bu\end{cases}$ $\quad x_1 = \phi - \pi/2$ (angle of pendulum), $x_2 = 0$

$$\sigma = [a\ \ 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = ax_1 + \underset{\dot{x}_1}{x_2} \Rightarrow \boxed{\dot{x}_1 = -ax_1}$$

$$\sigma = x_2 + ax_1 \Rightarrow \boxed{\dot{\sigma}} = -\theta\cos(x_1) + ax_2 + bu$$

Sliding mode control

- Design a control input that makes the state starting in $\sigma(x) = 0$ to stay there for all t. ($\sigma(x) = 0$ invariant)

Pendulum: $\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -\theta\cos(x_1) + bu\end{cases}$

$$\sigma = x_2 + ax_1 \Rightarrow \dot{\sigma} = -\theta\cos(x_1) + ax_2 + bu$$

$\dot{\sigma} = 0$ $\quad \sigma(x(0)) = 0$ $\quad \sigma(x(t)) = 0$ $\quad u = \frac{1}{b}\left(-ax_2 + \theta\cos(x_1)\right)$

u_{eq}.

Sliding mode control

- Use feedback linearization and design a control input that makes $\sigma(x) = 0$.

Pendulum: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\theta\cos(x_1) + bu \end{cases}$

$u_{\text{equiv}} = \frac{1}{b}\left(-ax_2 + \theta\cos(x_1)\right)$

Equivalent control

$\sigma = x_2 + ax_1 \Rightarrow \dot{\sigma} = -\theta\cos(x_1) + ax_2 + bu$

$u = u_{equiv} + v$

$V = \frac{1}{2}\sigma^2$

$\dot{V} = \sigma\dot{\sigma} = \sigma\left[\cancel{-\theta\cos(x_1) + ax_2 + bu_{equiv}} + bv\right]$

$\boxed{\text{Set } v = -K\sigma}$ $= -K\sigma^2$

Sliding mode control

- Design a discontinuous control input v as part of the control $u = u_{\text{equiv}} + v$ that drives $\sigma(x)$ to zero in finite time.

$\boxed{v = -K\text{sign}(\sigma)}$ $\quad \text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$

- Examine how $|\sigma(t)|$ changes over time

discontinuous feedback $v = -K\,\text{sgn}(\sigma)$: $\frac{d}{dt}|\sigma(t)| = \frac{d}{dt}\sqrt{\sigma^2} = \text{sign}(\sigma)\dot{\sigma} = -Kb$

$|\sigma(t)| = |\sigma(0)| - Kbt$ for $t \le \frac{\sigma(0)}{Kb}$

$\boxed{t_s = \frac{\sigma(0)}{Kb}}$

The solution σ not extended after this time instant.

continuous feedback $v = -K\sigma$: $\frac{d}{dt}\sigma = -Kb\sigma \Rightarrow \sigma(t) = e^{-Kbt}\sigma(0)$ exponential convergence

Sliding mode control

- Choose K such that $u = \hat{u}_{eq} + v$ drives $\sigma(x)$ to zero in finite time.

$\hat{u}_{eq}$ is designed by using some estimate $\hat{\theta}$ $\Rightarrow \hat{u}_{eq} = -\frac{a}{b}x_2 + \frac{\hat{\theta}}{b}\cos(x_1)$

$v = -K\text{sign}(\sigma)$

$\text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$

By choosing K sufficiently large → robustness to uncertainty

$\dot{\sigma} = -\theta\cos x_1 + ax_2 + b(\hat{u}_{eq} + v)$

$\dot{\sigma} = -(\theta - \hat{\theta})\cos x_1 - bK\,\text{sgn}(\sigma)$

$|\cos(x_1)| \le 1$

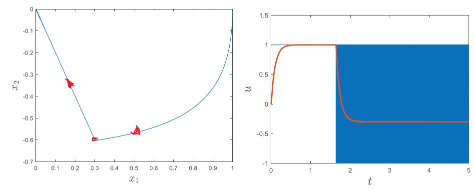
$|\theta - \hat{\theta}| \le \bar{\theta}$ "worst" case, most positive

$V = \frac{1}{2}\sigma^2 \Rightarrow \dot{V} = \sigma\dot{\sigma} = -bK|\sigma| - \sigma(\theta - \hat{\theta})\cos(x_1)$

$\le -bK|\sigma| + \boxed{\bar{\theta}|\sigma|}$

$\le -(bK - \bar{\theta})|\sigma| < 0$ for $K > \frac{\bar{\theta}}{b}$

Example



Pendulum: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.3\cos(x_1) - u \end{cases}$

Sliding mode controller: $\{u = \text{sign}(x_2 + 2x_1)$

Even higher uncertainty (no equivalent controller added)

- Chattering → wear on mechanical devices

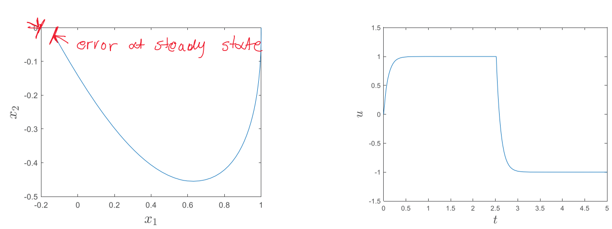
Robustness to uncertainty comes with the expense of high-frequency input signal

If it is smoothed there are steady state errors. see next slide

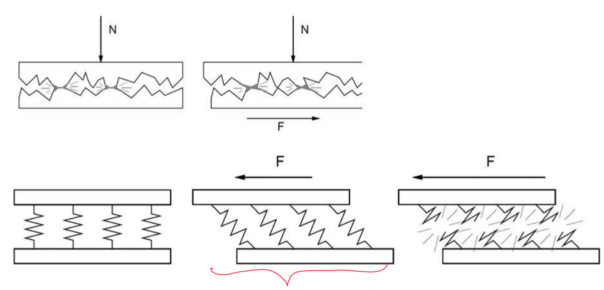
Example

- Continuous control $u = \tanh(x_2 + 2x_1)$

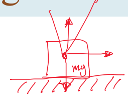
error at steady state

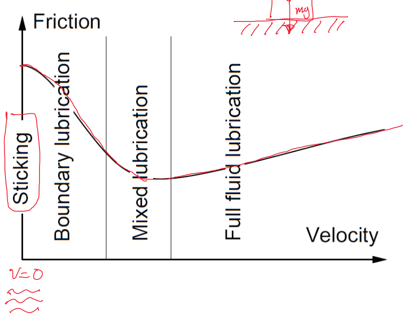


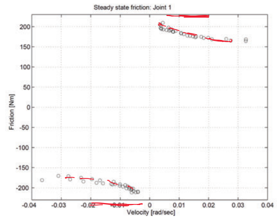
Friction



Lubrication regimes



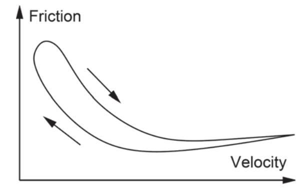




For low velocity: friction increases with decreasing velocity
Stribeck (1902)

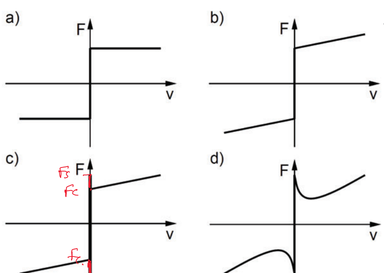
Hysteresis and friction

- Dynamics are important also outside sticking regime
- Hess and Soom (1990)
- Experiment with unidirectional motion $v(t) = v_0 + a\sin(\omega t)$
- Hysteresis effect!



Classical friction models

$m\ddot{x} + F(t) = F_e(t)$



$F_e(t) =$ external applied force , F_c, F_v, F_s constants

$$\boxed{F(t) = F_c \operatorname{sign} v(t)} \quad (a)$$

$$F(t) = \underbrace{F_v v(t)}_{\text{viscous}} + \underbrace{F_c \operatorname{sign} v(t)}_{\text{Coulomb}} \quad (b)$$

$$F(t) = \begin{cases} F_c \operatorname{sign} v(t) + F_v v(t) & v(t) \neq 0 \\ \underbrace{\max(\min(F_e(t), F_s), -F_s) \quad v(t) = 0}_{\text{equivalent expression}} \end{cases} \quad (c)$$

$$F(t) = \begin{cases} \left(F_c + (F_s - F_c)e^{-|\frac{v}{v_s}|^2}\right) \operatorname{sign} v(t) + F_v v(t) & v(t) \neq 0 \\ F_e(t) & v(t) = 0 \text{ and } |F_e(t)| < F_s \\ \operatorname{sign}(F_e) F_s & v(t) = 0 \text{ and } |F_e(t)| \geq F_s \end{cases} \quad (d)$$

Advanced Friction Models

- Karnopp model
- Armstrong's seven parameter model
- Dahl model
- Bristle model
- Reset integrator model
- Bliman and Sorine
- LuGre model (Lund-Grenoble)

See PhD-thesis by Henrik Olsson

https://lucris.lub.lu.se/ws/portalfiles/portal/4768278/8840259.pdf

Friction models with extended state

Dahl's model

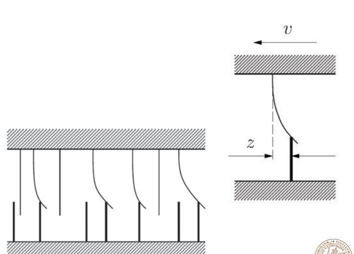
$$\dot{F}(t) = \sigma_0(v(t) - F_c|v(t)|)$$

LuGre model

$$\dot{z}(t) = v - \frac{|v|}{g(v)} z(t)$$

$$g(v) = \left(F_c + (F_s - F_c)e^{-|\frac{v}{v_s}|^2}\right)$$

$$F(t) = \underbrace{\sigma_0 z(t)}_{\text{stiffness of bristles}} + \underbrace{\sigma_1 \dot{z}}_{\text{damping}} + \underbrace{F_v v(t)}_{\text{viscous friction}}$$



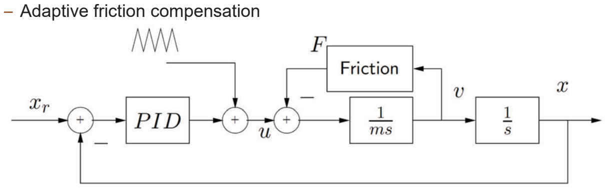
Contact between bristles

Friction and control

- Friction compensation
  - Lubrication
  - Dither
  - Integral action
  - Non-model based control
  - **Model-based friction compensation**
  - Adaptive friction compensation

High-frequency mechanical vibration: used to avoid sticking

Integral term compensate for slowly varying disturbances



Friction and control

- Friction compensation
  - Lubrication
  - Dither
  - Integral action
  - Non-model based control
  - **Model-based friction compensation**
  - Adaptive friction compensation

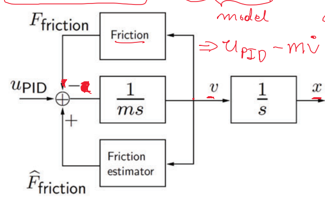
- To be useful for control the model should be:
  - sufficiently accurate,
  - suitable for simulation,
  - simple, few parameters to determine.
  - physical interpretations, insight
- Simple models should be preffed.
- If no stiction occurs the v=0-models are not needed.

Adaptive friction compensation

$\boxed{F_{\text{friction}} = a\,\text{sgn}(v)}$

$\underbrace{m\ddot{x} + F_{\text{friction}}}_{\text{model}} = \underbrace{u_{PID}}_{\text{controller}} + \hat{F}_{\text{friction}}$

$\Rightarrow u_{PID} - m\dot{v} = F - \hat{F}$



Assumption: v measurable.

$\boxed{\text{Friction estimator:}}$
$$\begin{cases} \dot{z} \xrightarrow{\text{update}} = ku_{\text{PID}}\text{sgn}(v) \\ \hat{a} = z - km|v| \\ \hat{F}_{\text{friction}} = \hat{a}\text{sgn}(v) \end{cases}$$

Result: $e = a - \hat{a} \to 0$ as $t \to \infty$:

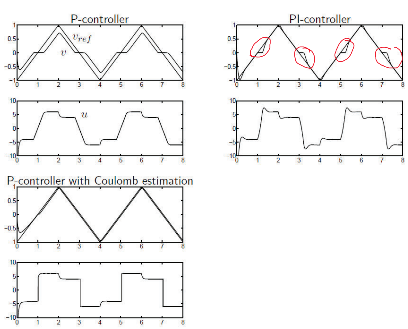
$\frac{d}{dt}\sqrt{v^2} = \text{sgn}(v)\,\dot{v}$

$$\begin{aligned} \frac{de}{dt} &= -\frac{d\hat{a}}{dt} = \frac{dz}{dt} - km\frac{d}{dt}|v| \\ &= ku_{\text{PID}}\text{sgn}(v) + km\dot{v}\text{sgn}(v) \\ &= k\text{sgn}(v)(u_{\text{PID}} - m\dot{v}) \\ &= -k\text{sgn}(v)(F - \hat{F}) = -k\text{sgn}(v)(a \cdot \text{sgn}(v) - \hat{a}\,\text{sgn}(v)) \\ &= -k(a - \hat{a}) \\ &= -ke \end{aligned}$$

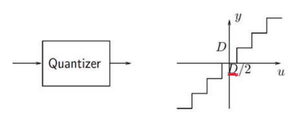
$\Rightarrow \boxed{e(t) = e(0)\,\exp(-kt)}$

$e(t) \to 0$

Velocity Control – Results

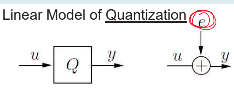


Quantization



- Digital signals have specific number of bits (accuracy and range of signals) (e.g. 8 bits, 64 bits)
- Quantization in A/D and D/A converters
- Quantization of parameters
- Roundoff, overflow, underflow in operations

Linear Model of Quantization



- e: Noise independent of u with rectangular distribution over the quantization size with rectangular distribution with variance

$$Var(e) = \int_{-\infty}^{+\infty} e^2 f_e\, de = \int_{-D/2}^{D/2} e^2 \frac{1}{D} de = \frac{D^2}{12}$$

- It works if quantization level is small compared to the variations in u

For prediction of limit cycles

- Use describing function analysis

Sliding mode control

- Choose K such that $u = v$ drives $\sigma(x)$ to zero in finite time.

$$u = -K\text{sign}(\sigma)$$
$$\text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$$

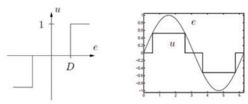
Sliding mode control

- Choose K such that $u = v$ drives $\sigma(x)$ to zero in finite time.

$$u = -K\text{sign}(\sigma)$$
$$\text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$$

Quantization: Describing function

- Recall the deadzone nonlinearity



$$Var(e) = \int_{-\infty}^{+\infty} e^2 f_e \, de = \int_{-D/2}^{D/2} e^2 \frac{1}{D} de = \frac{D^2}{12}$$

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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 14: Course summary

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Outline

- Examples

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Equilibrium points and Limit Cycles

$\dot{x} = f(x)$

- Find equilibrium points $f(x^*) = 0$
- Given a trajectory show that it is a limit cycle
- Classify equilibrium points
- Stability of limit cycle

Let $x^*(t)$ a periodic trajectory independent of initial condition

$\dot{x}^*(t) = f(x^*(t))$

$J_f = \left[\frac{\partial f}{\partial x}\right]_{x = x^*}$

Check the eigenvalues and classify according the classification of equilibriums in linear systems

$J_f = \left[\frac{\partial f}{\partial x}\right]_{x = x^*(t)}$

Now J_f is function of time

Stability can be checked by examining the eigenvalues of $J_f + J_f^T$

For limit cycle stability check you can also change coordinates to r, θ and then linearize the system

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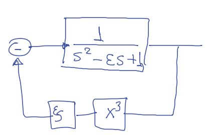
Existence of limit cycles – Describing Function Analysis

Consider a form of the van der Pol equation:

$$\ddot{x} + \varepsilon(3x^2 - 1)\dot{x} + x = 0$$

$\Rightarrow \ddot{x} - \varepsilon\dot{x} + x = \varepsilon \underbrace{3x^2\dot{x}}$

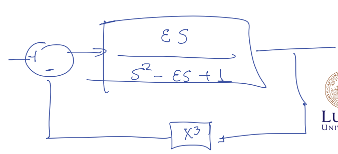
$\Rightarrow \ddot{x} - \varepsilon\dot{x} + x = -\varepsilon \frac{d}{dt}(x^3)$



Express the system as a transfer function $G(s)$ in the forward loop with a non-linear feedback though a static nonlinearity $h(x)$. Find explicitely $G(s)$ and $h(x)$.

The TF has two poles with positive real part.

$\# OL_{unstable} = 2$



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Existence of limit cycles – Describing Function Analysis

Consider a form of the van der Pol equation:

$$\ddot{x} + \varepsilon(3x^2 - 1)\dot{x} + x = 0$$

$\# CL_{unstable} = \# OL_{unstable} - \#$ encirclements

if "out of" the circle #encirclements = 0

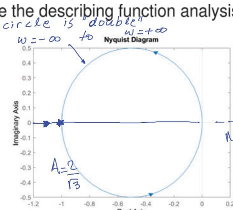
$\# CL_{unstable} = 2$ (unstable)

if "in the" circle # encirclements = 2 (*)

Thia is in fact the van der Pol oscillator that generates limit cycles. $\# CL_{unstable} = 0$ (stable)

Use the describing function analysis to examine the existence of a limit cycle.

(*) The circle is "double" from $\omega = -\infty$ to $\omega = +\infty$



$-\frac{1}{N(A)}$

$$\boxed{N(A) = \frac{3A^2}{4}} \longrightarrow -\frac{1}{N(A)} = -\frac{4}{3A^2} = -1 \Rightarrow \boxed{A = \frac{2}{\sqrt{3}}}$$

If A increases $-\frac{1}{N(A)}$ moves towards origin.

If $A^* = \frac{2}{\sqrt{3}}$ disturbed to the left (decreased) then unstable (out of the circle) and returns back

If $A^* = \frac{2}{\sqrt{3}}$ disturbed to the right (increased) then stable (in the circle) and returns back (decreases).

The predicted limit cycle is stable

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Lyapunov stability analysis

1. $V > 0$ p.d. $V(x)$ if $x = 0$ for $x \in \Omega$ $\Omega \equiv \mathbb{R}^n$
$V(x) > 0$ if $x \neq 0$

def. $[\dot{V} < 0 \sim -\dot{V} > 0$ p.d.$]$

V radially unbounded.

2. $-\dot{V} > 0$ p.d. for $x \in \Omega$. $\Omega \equiv \mathbb{R}^n$

$\Rightarrow$ asymptotic stability (local)

global.

If $\dot{V} \leq 0$ then stable equilibrium point.

La Salle

$$\left.\begin{array}{l}\dot{V} = 0 \\ \dot{x} = f(x)\end{array}\right\} \Rightarrow \boxed{x = 0}$$

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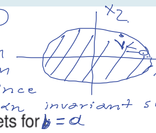
Lyapunov stability analysis

$$\begin{cases}\dot{x}_1 = \frac{1}{a}x_2 \\ \dot{x}_2 = -bx_1 - x_2(1 - a^2x_1^2 - x_2^2)\end{cases}$$



(*) Here the actual region of attraction is $V < 1$ since $V = 1$ is an invariant set [an unstable limit cycle]

Show that the origin and the ellipse $a^2x_1^2 + x_2^2 = 1$ are invariant sets for $b = a$

Choose a Lyapunov function to show that the origin is a locally asymptotical stable equilibium point.

Inspired by the first question we choose $V = a^2x_1^2 + x_2^2 > 0$ p.d.

What is the region of attraction?

$\dot{V} = -x_2^2(1 - a^2x_1^2 - x_2^2)$

If $x \in \Omega_1 \equiv \{x \in \mathbb{R}^2, V < 1\}$ in the ellipse

$\dot{V} = -\varepsilon x_2^2 \leq 0$ (Not negative definite)

A region of the form $\Omega_1 = \{V < 1\}$ (*) is an estimate of the RA.

LaSalle In Ω_1 $M = \{\text{origin}\}$ $x \to 0$ for $x \in \Omega_1$

(take $x_2 = 0 \Rightarrow x_2 \equiv 0 \Rightarrow x_1 = 0$)

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Lyapunov stability analysis

$$\begin{cases}\dot{x}_1 = \frac{1}{a}x_2 \\ \dot{x}_2 = -bx_1 - x_2(1 - a^2x_1^2 - x_2^2)\end{cases}$$

$(0,0)$

Show that the origin and the ellipse $a^2x_1^2 + x_2^2 = 1$ are invariant sets for $b = a$

origin

For $x_1 = x_2 = 0$ we get $\dot{x}_1 = 0$, $\dot{x}_2 = 0$

ellipse

$\sigma = a^2x_1^2 + x_2^2 - 1$

for $b = a$

$\dot{\sigma} = \frac{d\sigma}{dt} = 2a^2x_1\dot{x}_1 + 2x_2\dot{x}_2 = \cancel{2ax_1x_2} - \cancel{2bx_2x_1} + x_2^2(a^2x_1^2 + x_2^2 - 1)$

$= x_2^2(a^2x_1^2 + x_2^2 - 1)$

For $\underbrace{\sigma = 0 \text{ we get } \dot{\sigma} = 0}_{\text{similar to what we did for equilibrium point (origin)}}$ i.e. $\sigma(t) = \sigma(0) = 0$ $\forall t$

invariant

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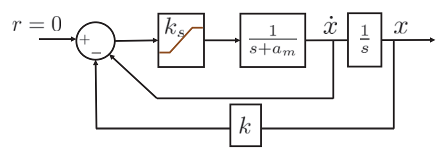
BIBO – Circle Criterion etc.

A motor is controlled using position and velocity feedback combined in a P-controller with gain k_s. A gain k is used as a weight for the position feedback. The input signal is saturated due to input torque limitations. In particular $u = -k_s(\dot{x} + kx)$ if $-1 < u < 1$, otherwise u is saturated to -1 or $+1$ depending on the signum of $\dot{x} + kx$. Use the circle criterion to study the asymptotic stability of the origin.



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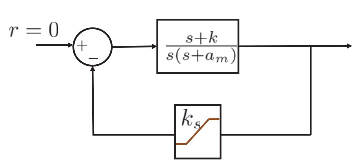

$$G(s) = \frac{s+k}{s(s+a_m)}$$

$$G(j\omega) = \frac{j\omega + k}{-\omega^2 + j\omega a_m} \Rightarrow \mathrm{Re}[G(j\omega)] = \frac{a_m - k}{\omega^2 + a_m^2}$$

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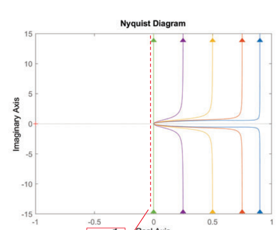
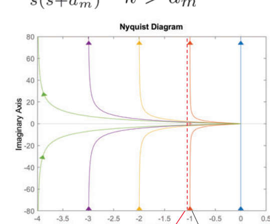
Circle Criterion

$a_m > k$ **Nyquist** $G(s) = \frac{s+k}{s(s+a_m)}$ $k > a_m$





$-\frac{1}{k_s}$

$-\frac{1}{k_s}$ $\frac{a_m - k}{a_m^2}$

You could check the conditions for strictly positive real functions and get the result without a Nyquist graph.

k_s arbitrarily large

$$k_s < \frac{a_m^2}{k - a_m}$$

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Nonlinear Control Design

(b) V p.d for $\alpha, b > 0$ positive constant

$$\begin{cases} \dot{x}_1 = -x_1 + \theta x_1^2 x_2 \\ \dot{x}_2 = u \end{cases}$$, θ positive constant

$$\dot{V} = a x_1 \dot{x}_1 + b x_2 \dot{x}_2 =$$
$$= \underline{-a x_1^2} + \underbrace{a\theta x_1^3 x_2} + b x_2 u$$
$$u = -\frac{\alpha}{b}\theta x_1^3 - k x_2$$

(a) Assume that x_1 is not measurable and thus cannot be used for control. Use the function $V = \frac{1}{2}(ax_1^2 + bx_2^2)$ to show that linear feedback of x_2 can achieve local asymptotic stability of the origin.

$$\dot{V} = -\alpha x_1^2 - kb x_2^2$$
$$\dot{V} < 0$$

(b) Assume full state feedback can be used. Derive a controller that can achieve global asymptotic stability of the origin? $\boxed{\text{Is the stabilty property exponential?} *}$

(c) α, b are free parameters

(c) Is your controller robust to uncertainty in the parameter θ? choose $\frac{\alpha}{b} = \frac{\varepsilon}{\theta}$ where ε α positive constant

$*$ $\min(\alpha, b)\|x\|^2 \leq V \leq \max(\alpha, b)\|x\|^2$

$\dot{V} \leq -\min(\alpha, kb)\|x\|^2$

$\Rightarrow$ Exponential stability

Then $u = -\varepsilon x_1^3 - k x_2$ independent of θ

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