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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 14: Course summary

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Outline

- Examples

Equilibrium points and Limit Cycles

$\dot{x} = f(x)$

- Find equilibrium points $f(x^*) = 0$
- Given a trajectory show that it is a limit cycle

  Let $x^*(t)$ a periodic trajectory independent of initial condition

  $$\dot{x}^*(t) = f(x^*(t))$$
- Classify equilibrium points

  $$J_f = \left[\frac{\partial f}{\partial x}\right]_{x=x^*}$$

  check the eigenvalues and classify according the classification of equilibriums in linear systems
- Stability of limit cycle

$$J_f = \left[\frac{\partial f}{\partial x}\right]_{x=x^*(t)}$$

Now J_f is function of time

Stability can be checked by examining the eigenvalues of $J_f + J_f^T$

For limit cycle stability check you can also change coordinates to r, θ and then linearize the system

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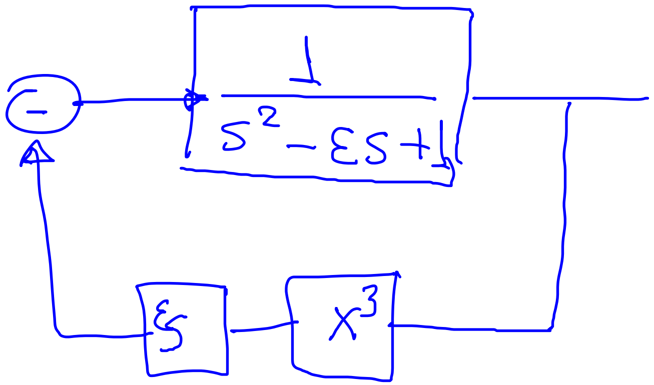
Existence of limit cycles – Describing Function Analysis

Consider a form of the van der Pol equation:

$$\ddot{x} + \varepsilon(3x^2 - 1)\dot{x} + x = 0$$

$$\Rightarrow \ddot{x} - \varepsilon\dot{x} + x = \varepsilon\, \underbrace{3x^2\dot{x}}$$

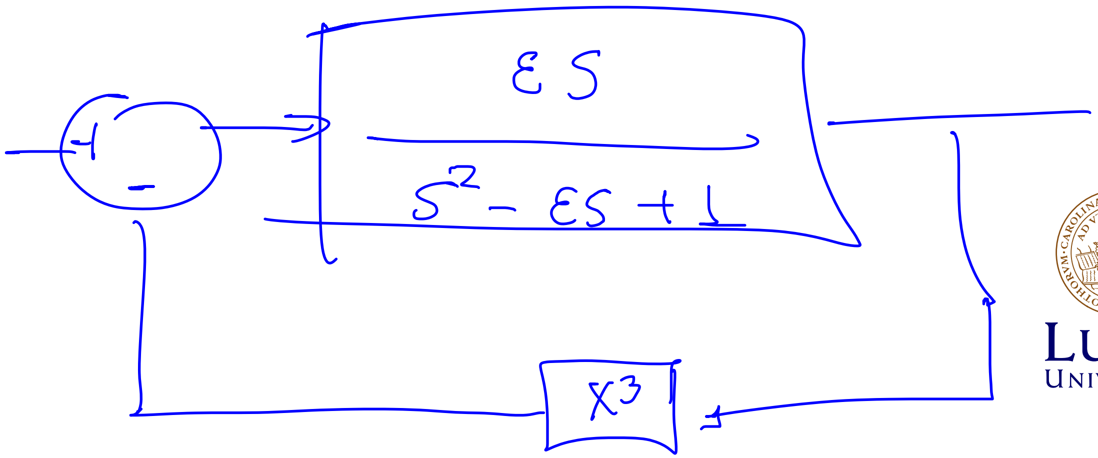
$$\Rightarrow \ddot{x} - \varepsilon\dot{x} + x = -\varepsilon \frac{d}{dt}(x^3)$$



Express the system as a transfer function $G(s)$ in the forward loop with a non-linear feedback though a static nonlinearity $h(x)$. Find explicitely $G(s)$ and $h(x)$.

The TF has two poles with positive real part.

OL unstable = 2



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Existence of limit cycles – Describing Function Analysis

Consider a form of the van der Pol equation:

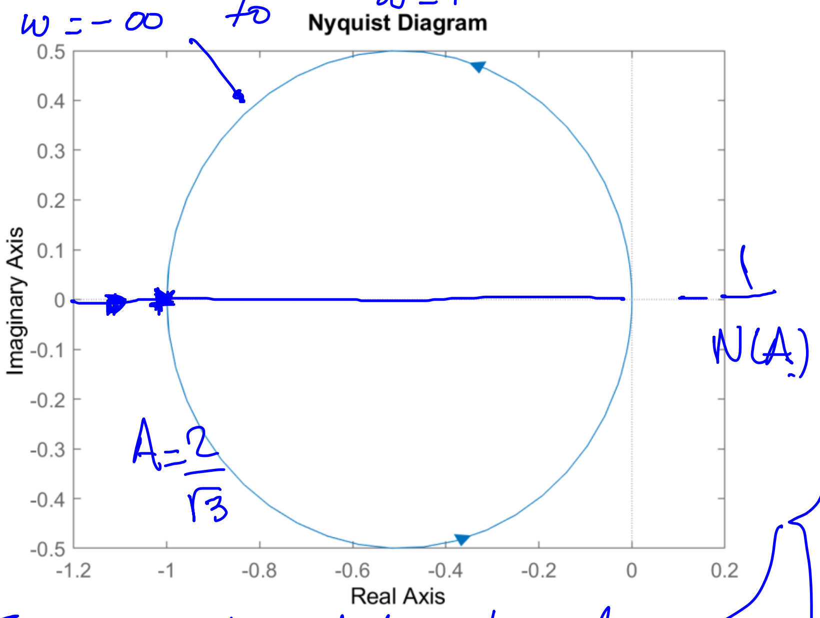
$$\ddot{x} + \varepsilon(3x^2 - 1)\dot{x} + x = 0$$

Thia is in fact the van der Pol oscillator that generates limit cycles.

Use the describing function analysis to examine the existence of a limit cycle.

#CL unstable = #OL unstable − # encirclements
if "out of" the circle #encirclements = 0
#CL unstable = 2 (unstable)
if "in the" circle # encirclements = 2 *
#CL unstable = 0 (stable)

* The circle is "double" from $\omega = -\infty$ to $\omega = +\infty$



The predicted limit cycle is stable

$$\boxed{N(A) = \frac{3A^2}{4}} \longrightarrow -\frac{1}{N(A)} = -\frac{4}{3A^2} = -1 \Rightarrow \boxed{A = \frac{2}{\sqrt{3}}}$$

If A increases $-\frac{1}{N(A)}$ moves towards origin.

If $A^* = \frac{2}{\sqrt{3}}$ disturbed to the left (decreased) then unstable (out of the circle) and returns back

If $A^* = \frac{2}{\sqrt{3}}$ disturbed to the right (increased) then stable (in the circle) and returns back (decreases).

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Lyapunov stability analysis

1. $V > 0$ p.d. $V(x)$ if $x = 0$ for $x \in \Omega$, $\Omega \in \mathbb{R}^n$

$V(x) > 0$ if $x \neq 0$

def. $[\dot{V} < 0 \sim -\dot{V} > 0$ p.d.$]$

V radially unbounded

2. $-\dot{V} > 0$ p.d. for $x \in \Omega$. $\Omega = \mathbb{R}^n$.

$\Rightarrow$ asymptotic stability (local)

global.

If $\dot{V} \leq 0$ then stable equilibrium point.

La Salle

$$\left.\begin{array}{l}\dot{V} = 0 \\ \dot{x} = f(x)\end{array}\right\} \Rightarrow \boxed{x = 0}$$

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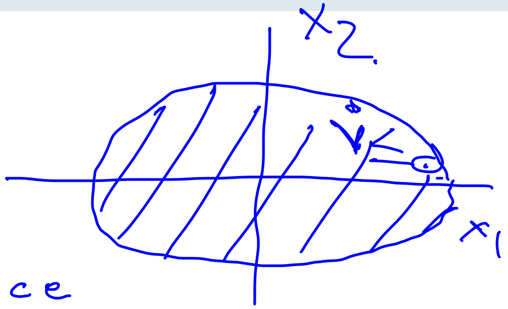
Lyapunov stability analysis

$$\begin{cases} \dot{x}_1 = \frac{1}{a}x_2 \\ \dot{x}_2 = -bx_1 - x_2(1 - a^2x_1^2 - x_2^2) \end{cases}$$

⊛ Here the actual region of attraction is $V<1$ since $V=1$ is an invariant set [an unstable limit cycle]



Show that the origin and the ellipse $a^2x_1^2 + x_2^2 = 1$ are invariant sets for $b = a$

Choose a Lyapunov function to show that the origin is a locally asymptotical stable equilibium point.

Inspired by the first question we choose $V = a^2x_1^2 + x_2^2 > 0$ p.d.

What is the region of attraction?

$\dot V = -x_2^2(1 - a^2x_1^2 - x_2^2)$

A region of the form $\Omega_1 = \{V < 1\}$ ⊛ is an estimate of the RA.

If $x \in \Omega_1 \equiv \{x \in \mathbb{R}^2, V < 1\}$ in the ellipse

$\dot V = -\varepsilon x_2^2 \leq 0$ (Not negative definite)

LaSalle In Ω_1 $M = \{\text{origin}\}$ $x \to 0$ for $x \in \Omega_1$

(take $x_2 = 0 \Rightarrow \dot x_2 \equiv 0 \Rightarrow x_1 = 0$)

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Lyapunov stability analysis

$$\begin{cases} \dot{x}_1 = \frac{1}{a} x_2 \\ \dot{x}_2 = -b x_1 - x_2(1 - a^2 x_1^2 - x_2^2) \end{cases}$$

$(0,0)$

Show that the origin and the ellipse $a^2 x_1^2 + x_2^2 = 1$ are invariant sets for $b = a$

Origin

For $x_1 = x_2 = 0$ we get $\dot{x}_1 = 0$, $\dot{x}_2 = 0$

ellipse

$\sigma = a^2 x_1^2 + x_2^2 - 1$

for $b = a$

$\dot{\sigma} = \frac{d\sigma}{dt} = 2a^2 x_1 \dot{x}_1 + 2 x_2 \dot{x}_2 = \cancel{2 a x_1 x_2} - \cancel{2 b x_2 x_1} + x_2^2 (a^2 x_1^2 + x_2^2 - 1)$

$= x_2^2 (a^2 x_1^2 + x_2^2 - 1)$

For $\underbrace{\sigma = 0 \text{ we get } \dot{\sigma} = 0}_{\text{similar to what we did for equilibrium point (origin)}}$ i.e. $\sigma(t) = \sigma(0) = 0 \ \forall t$

invariant

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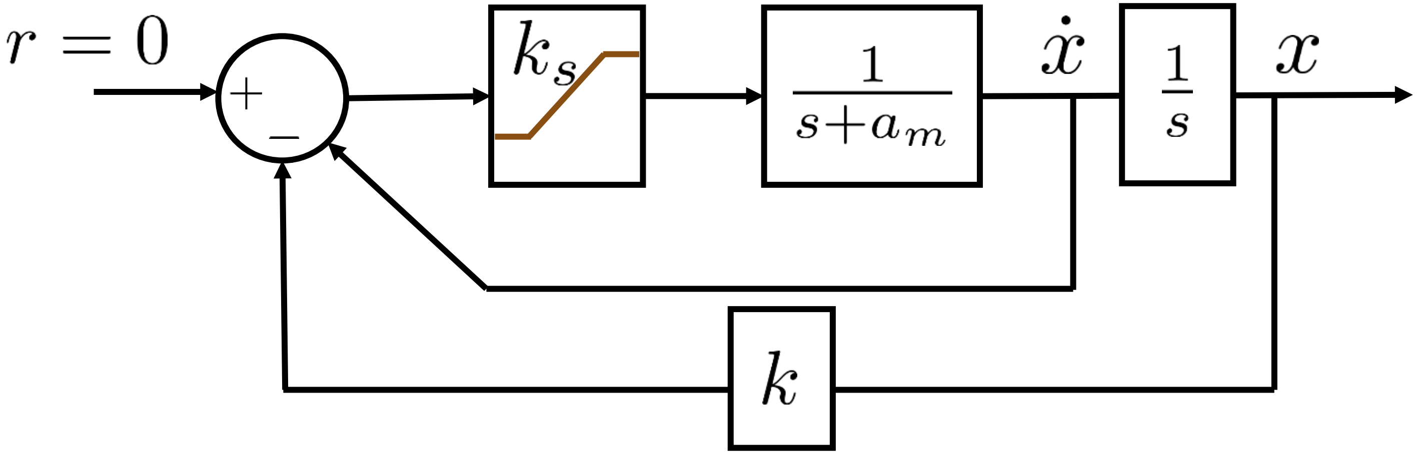
BIBO – Circle Criterion etc.

A motor is controlled using position and velocity feedback combined in a P-controller with gain k_s. A gain k is used as a weight for the position feedback. The input signal is saturated due to input torque limitations. In particular $u = -k_s(\dot{x} + kx)$ if $-1 < u < 1$, otherwise u is saturated to -1 or $+1$ depending on the signum of $\dot{x} + kx$. Use the circle criterion to study the asymptotic stability of the origin.



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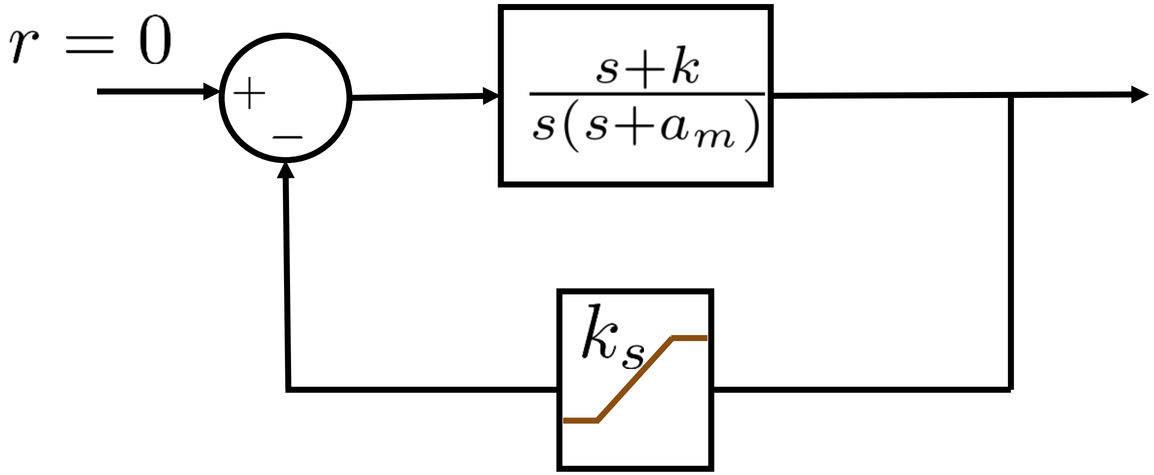
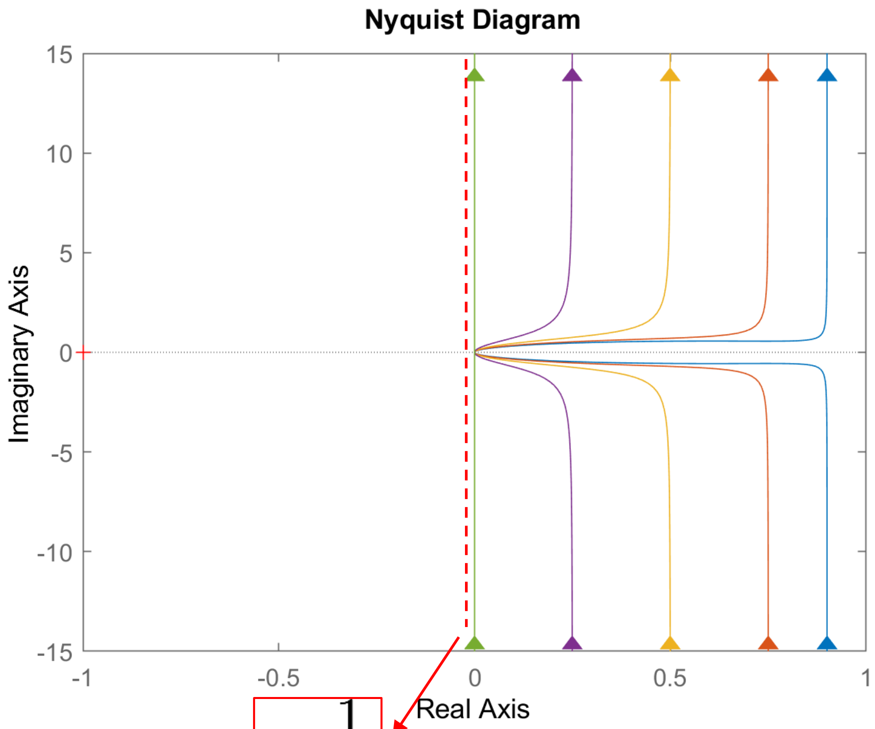


$$G(s) = \frac{s+k}{s(s+a_m)}$$

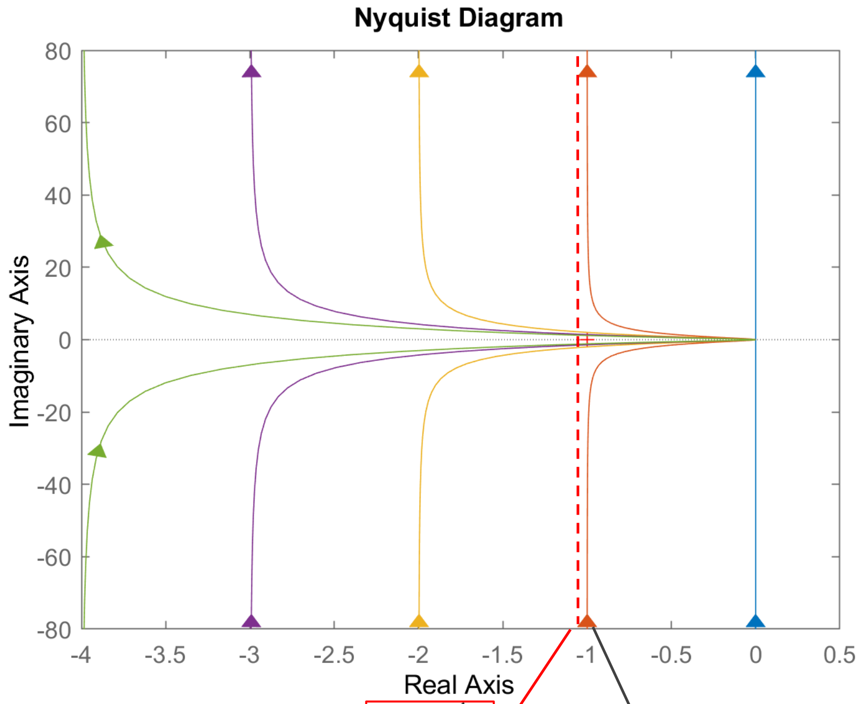
$$G(j\omega) = \frac{j\omega+k}{-\omega^2+j\omega a_m} \Rightarrow \mathrm{Re}[G(j\omega)] = \frac{a_m-k}{\omega^2+a_m^2}$$

Circle Criterion

$a_m > k$ **Nyquist** $G(s) = \frac{s+k}{s(s+a_m)}$ $k > a_m$



$\boxed{-\frac{1}{k_s}}$



$\boxed{-\frac{1}{k_s}}$ $\boxed{\frac{a_m - k}{a_m^2}}$

You could check the conditions for strictly positive real functions and get the result without a Nyquist graph.

k_s arbitrarily large

$$k_s < \frac{a_m^2}{k - a_m}$$

Nonlinear Control Design

(b) V p.d for $\alpha, b > 0$ positive constant

$$\begin{cases} \dot{x}_1 = -x_1 + \theta x_1^2 x_2 \\ \dot{x}_2 = u \end{cases}$$

, θ positive constant

$\dot{V} = a x_1 \dot{x}_1 + b x_2 \dot{x}_2 =$

$= \underline{-a x_1^2} + \underbrace{a\theta x_1^3 x_2} + b \underline{x_2} u$

$u = -\frac{a}{b}\theta x_1^3 - k x_2$

(a) Assume that x_1 is not measurable and thus cannot be used for control. Use the function $V = \frac{1}{2}(ax_1^2 + bx_2^2)$ to show that linear feedback of x_2 can achieve local asymptotic stability of the origin.

$\dot{V} = -a x_1^2 - k b x_2^2$

$\dot{V} < 0$

(b) Assume full state feedback can be used. Derive a controller that can achieve global asymptotic stability of the origin? $\boxed{\text{Is the stabilty property exponential? *}}$

(c) α, b are free parameters

(c) Is your controller robust to uncertainty in the parameter θ? choose $\frac{\alpha}{b} = \frac{\varepsilon}{\theta}$ where ε & α positive constant

Then $u = -\varepsilon x_1^3 - k x_2$ independent of θ

* $\min(a, b)\|x\|^2 \le V \le \max(\alpha, b)\|x\|^2$

$\dot{V} \le -\min(\alpha, kb)\|x\|^2$

$\Rightarrow$ Exponential Stability

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