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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 13: Control Systems with discontinuities

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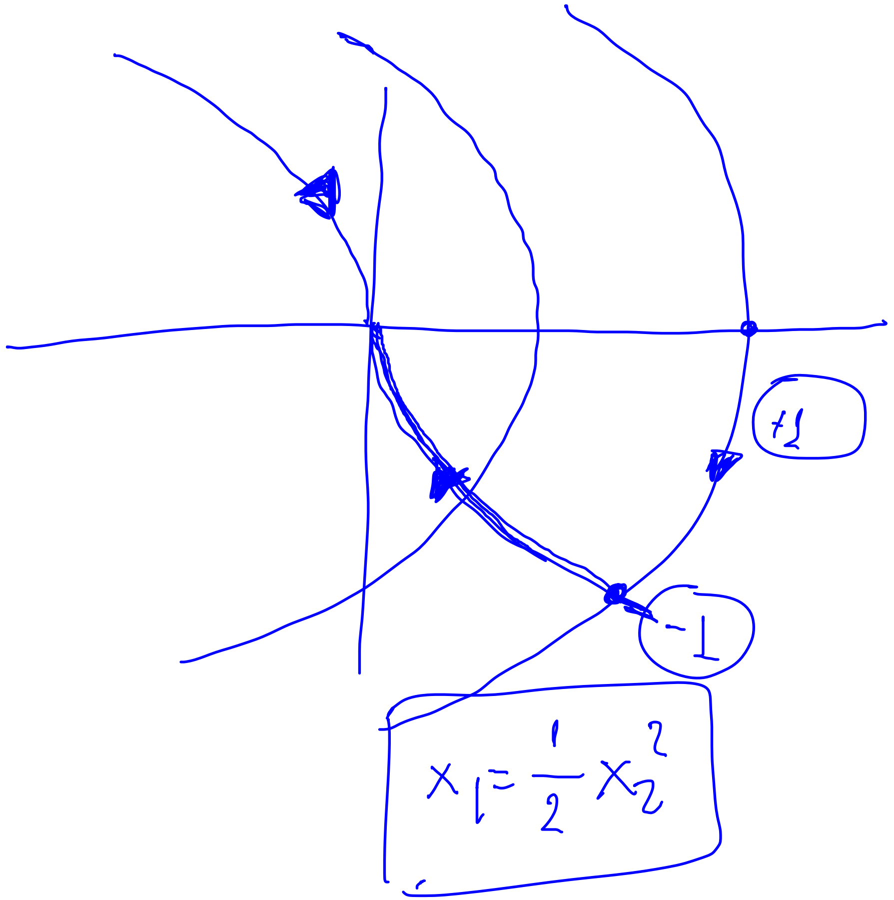
Outline

- ~~Time optimal control~~
- Sliding mode control
- Friction



Sliding set

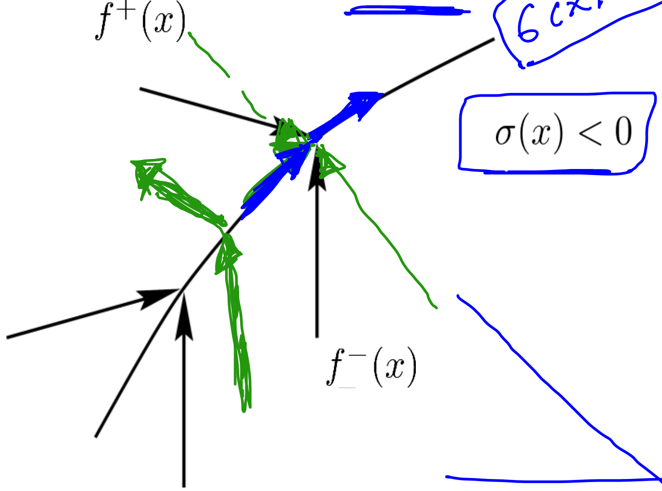
$$\dot{x} = \begin{cases} f^+(x) & \sigma(x) > 0 \\ f^-(x) & \sigma(x) < 0 \end{cases}$$

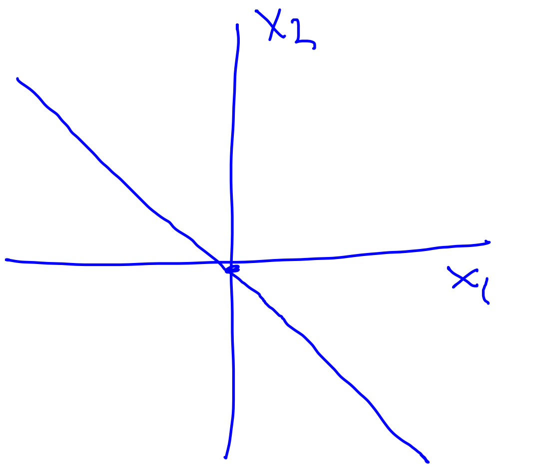
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **Sliding set:**

  $\sigma(x) = 0$

  $$\left.\begin{aligned} \frac{\partial\sigma}{\partial x} f^+ < 0 \\ \frac{\partial\sigma}{\partial x} f^- > 0 \end{aligned}\right\} \quad f^+ \text{ and } f^- \text{ point towards } \sigma(x) = 0$$





$x_1 + \alpha x_2 = 0$

- Once the trajectory hits the switching surface S, it cannot leave it because the vector fields on both sides point towards the surface

- If f^+ and f^- point “in the same direction” on both sides of the set $\sigma(x) = 0$, then the solution curves will just pass through and this region will not belong to the sliding set.

- If $f^+(x) = f^-(x)$ on $\sigma(x)$ then the sliding set is actually a set of equilibrium points

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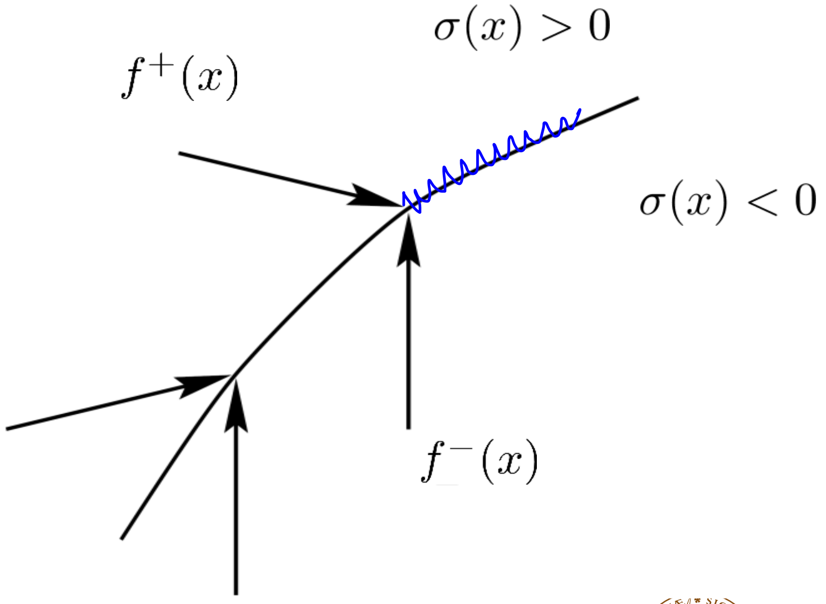
Sliding set

- The general behavior for the solution is to slide on $\sigma(x)$ (sliding mode.)
- The sliding motion can be described by noting that there is a unique convex combination of $f^+(x)$ and $f^-(x)$ that is tangent to $\sigma(x)$ at the point x.
  - **Sliding dynamics:**

$$\dot{x} = \alpha(x) f^+(x) + (1-\alpha(x)) f^-(x),$$

where $\alpha(x)$ is obtained from

$$\boxed{0 = \frac{d\sigma}{dt}} = \frac{\partial \sigma}{\partial x} \cdot \dot{x}$$
$$= \frac{\partial \sigma}{\partial x} f^-(x) + \alpha(x) \frac{\partial \sigma}{\partial x} \left[f^+(x) - f^-(x) \right].$$



- The fast switches will give rise to average dynamics that slide along the set where $\sigma(x) = 0$.

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Example

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u = Ax + Bu$$

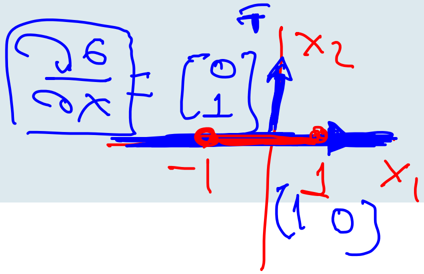
$$u = -\text{sgn}\sigma(x) = -\text{sgn}x_2 = -\text{sgn}(Cx)$$

which means that

$$\boxed{\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}}$$

Determine the *sliding set* and the *sliding dynamics*.

Example: Sliding set



$$\dot{x}_1 = -x_2 + u = -x_2 - \text{sgn}(x_2)$$
$$\dot{x}_2 = x_1 - x_2 + u = x_1 - x_2 - \text{sgn}(x_2)$$

σ(x)

$$\sigma(x) = 0 \qquad f^+ = \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} \qquad f^- = \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix}$$

σ(x) = x2

$$\sigma(x) = 0 \qquad\qquad x_2 = 0$$

$[0 \quad 1]$

$$\frac{\partial \sigma}{\partial x} f^+ < 0 \qquad\qquad x_1 - x_2 - 1 < 0$$

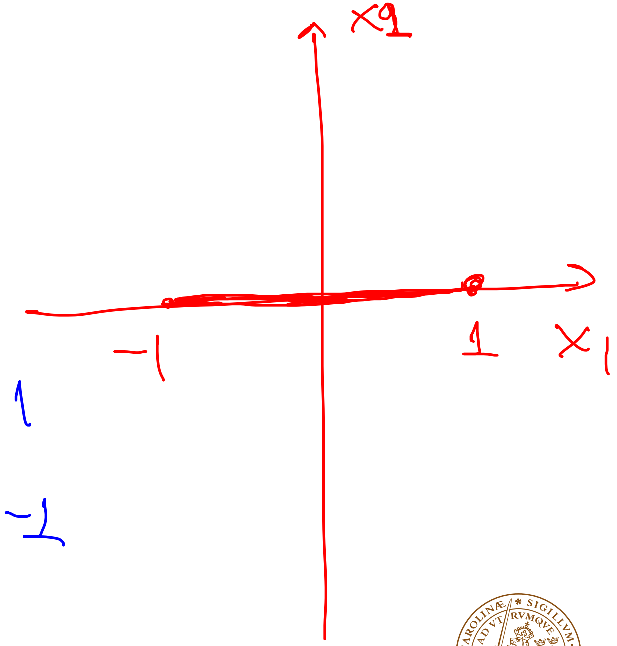
$$\frac{\partial \sigma}{\partial x} f^- > 0 \qquad\qquad x_1 - x_2 + 1 > 0$$

$\frac{\partial \sigma}{\partial x_1} = 0$

$\frac{\partial \sigma}{\partial x_2} = 1$

$x_1 < 1$

$x_1 > -1$



We thus have the sliding set $\{-1 < x_1 < 1,\ x_2 = 0\}$

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Example: Sliding dynamics

The *sliding dynamics* are the given by

$$\dot{x} = \alpha f^{+} + (1-\alpha) f^{-} \quad \text{sliding dynamics}$$
$$0 = \frac{\partial \sigma}{\partial x} f^{-}(x) + \alpha(x) \frac{\partial \sigma}{\partial x}\left[f^{+}(x) - f^{-}(x)\right]$$

On the sliding set $\{-1 < x < 1,\ x_2 = 0\}$, this gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \alpha \begin{bmatrix} -x_2 - 1 \\ x_1 - x_2 - 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} -x_2 + 1 \\ x_1 - x_2 + 1 \end{bmatrix}$$
$$0 = x_1 - x_2 + 1 - 2\alpha$$

Eliminating α gives $\begin{cases} \boxed{\dot{x}_1 = -x_1} \\ \dot{x}_2 = 0 \end{cases}$ Hence, any initial condition the sliding set will give exponential convergence to $x_1 = x_2 = 0$.

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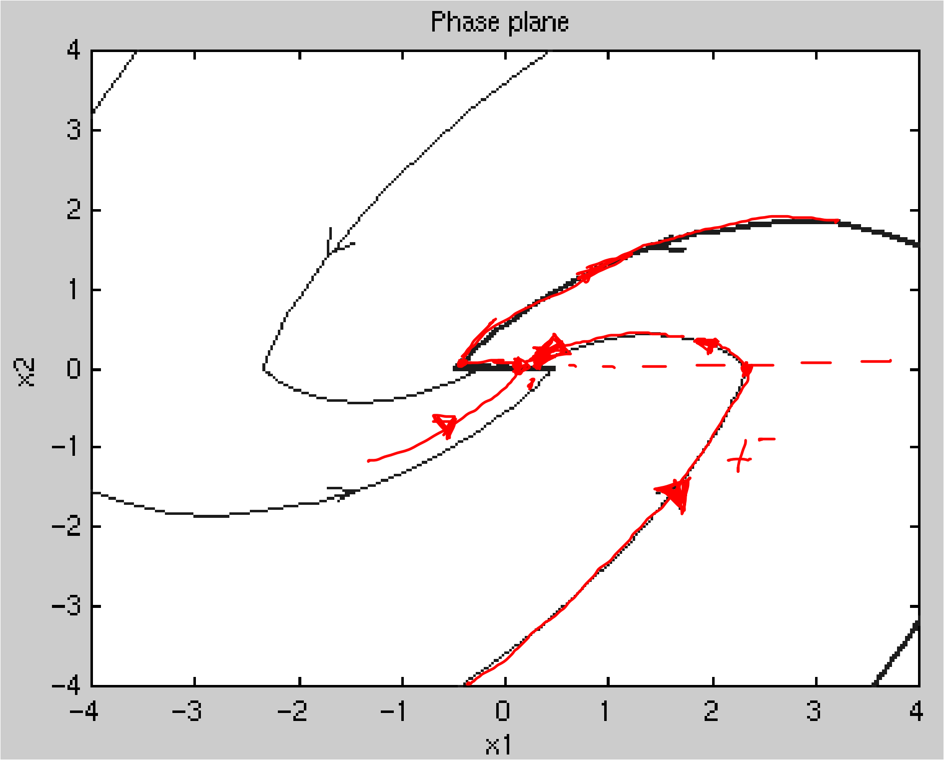

The dynamics along the sliding set in $\sigma(x) = 0$ can also be obtained by finding $u = u_{\text{eq}} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.

u_{eq} is called the **equivalent control**.

Sliding mode control

- Define an output $y = \sigma(x)$ such that the relative degree of the input-output relationship is 1 and the origin for the the system $\sigma(x) = 0$ is stable.

Pendulum: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\theta\cos(x_1) + bu \end{cases}$

angle of pendulum

$x_1 = \phi - \pi/2$

$x_2 = 0$

$$\sigma = [a \;\; 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1 + \underset{\dot{x}_1}{\underset{\parallel}{x_2}} \Rightarrow \boxed{\dot{x}_1 = -ax_1}$$

$$\sigma = x_2 + ax_1 \Rightarrow \boxed{\dot{\sigma}} = \underbrace{-\theta\cos(x_1) + ax_2}\, + bu$$

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Sliding mode control

- Design a control input that makes the state starting in $\sigma(x) = 0$ to stay there for all t. ($\sigma(x) = 0$ invariant)

Pendulum: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\theta\cos(x_1) + bu \end{cases}$

$$\sigma = x_2 + ax_1 \Rightarrow \dot{\sigma} = -\theta\cos(x_1) + ax_2 + bu$$

$\dot{\sigma} = 0$ $\quad \sigma(x(0)) = 0$

$\sigma(x(t)) = 0$

$u = \frac{1}{b}\left(-ax_2 + \theta\cos(x_1)\right)$

u_{eq}.

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Sliding mode control

- Use feedback linearization and design a control input that makes $\sigma(x) = 0$.

$u = u_{equiv} + v$

Pendulum: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\theta\cos(x_1) + bu \end{cases}$

$V = \frac{1}{2}\sigma^2$

$\dot{V} = \cancel{2}\frac{1}{\cancel{2}}\sigma\dot{\sigma} = \sigma\,[-\theta\cos(x_1) + a x_2 + b u_{equiv} + bv\,]$

$$u_{\text{equiv}} = \frac{1}{b}\left(-ax_2 + \theta\cos(x_1)\right)$$

Equivalent control

$\boxed{\text{Set } v = -k\sigma} \;=\; -k\sigma^2$

$$\sigma = x_2 + ax_1 \Rightarrow \dot{\sigma} = -\theta\cos(x_1) + ax_2 + bu$$

Sliding mode control

- Design a discontinuous control input v as part of the control $u = u_{\text{equiv}} + v$ that drives $\sigma(x)$ to zero in finite time.

$$\boxed{v = -K\text{sign}(\sigma)} \qquad \text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$$

- Examine how $|\sigma(t)|$ changes over time

discontinuous feedback $v = -K\,\text{sgn}(\sigma)$

$$\frac{d}{dt}|\sigma(t)| = \frac{d}{dt}\sqrt{\sigma^2} = \text{sign}(\sigma)\,\dot{\sigma} = -Kb$$

$|\sigma(t)| = |\sigma(0)| - Kbt$ for $t \le \frac{\sigma(0)}{Kb}$

$$\boxed{t_s = \frac{\sigma(0)}{Kb}}$$

The solution is not extended after this time instant

Continuous feedback $v = -K\sigma$

$$\frac{d}{dt}\sigma = -Kb\,\sigma \Rightarrow \sigma(t) = e^{-Kbt}\sigma(0)$$

exponential convergence

Sliding mode control

- Choose K such that $u = \hat{u}_{eq} + v$ drives $\sigma(x)$ to zero in finite time.

$\hat{u}_{eq}$ is designed by using some estimate $\hat{\theta}$ $\Rightarrow u_{eq} = -\frac{a}{b}x_2 + \frac{\hat{\theta}}{b}\cos(x_1)$

$$v = -K\,\mathrm{sign}(\sigma)$$

$$\mathrm{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$$

By choosing K sufficiently large robustness to uncertainty

$$\dot{\sigma} = -\theta\cos x_1 + a x_2 + b(\hat{u}_{eq} + v)$$

$$\dot{\sigma} = -(\theta - \hat{\theta})\cos x_1 - bK\,\mathrm{sgn}(\sigma)$$

$|\cos(x_1)| \le 1$

$|\theta - \hat{\theta}| \le \bar{\theta}$

"worst" case

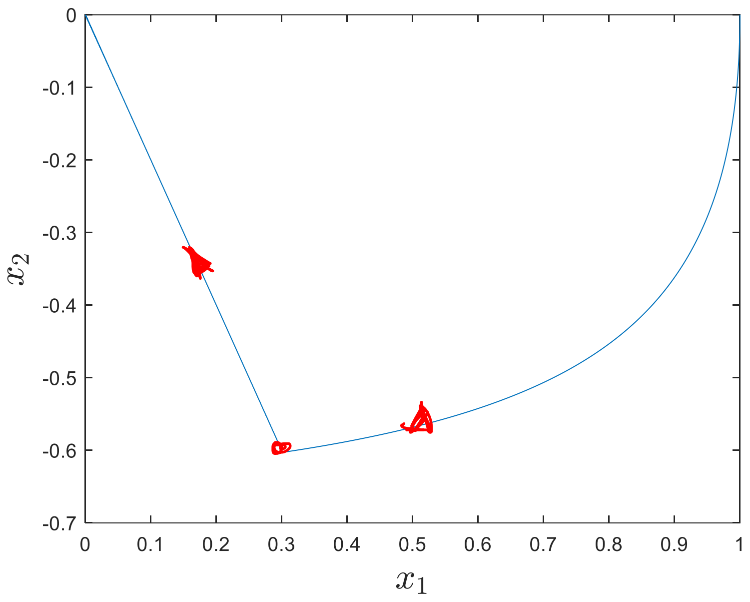
most positive

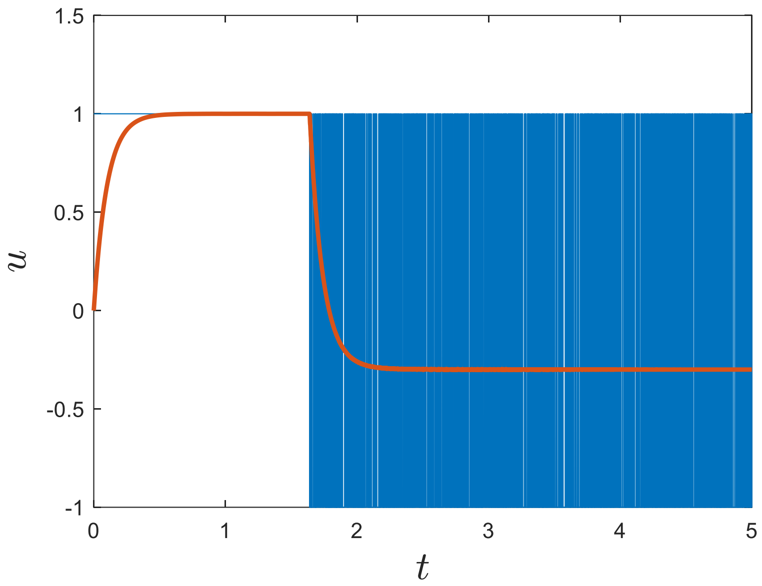
$V = \frac{1}{2}\sigma^2 \quad \Rightarrow \quad \dot{V} = \sigma\dot{\sigma} = -bK|\sigma| - \sigma(\theta - \hat{\theta})\cos(x_1)$

$$\le -bK|\sigma| + \boxed{\bar{\theta}\,|\sigma|}$$

$$\le -(bK - \bar{\theta})|\sigma| < 0 \quad \text{for } K > \frac{\bar{\theta}}{b}$$

Example





- *Chattering → wear on mechanical devices*

Robustness to uncertainty comes with the expense of high-frequency input signal

If it is smoothed ~~there~~ there are steady state errors. see next slide

Pendulum: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.3\cos(x_1) - u \end{cases}$

Even higher uncertainty

Sliding mode controller: $\begin{cases} u = \text{sign}(x_2 + 2x_1) \end{cases}$

(no equivalent controller added)

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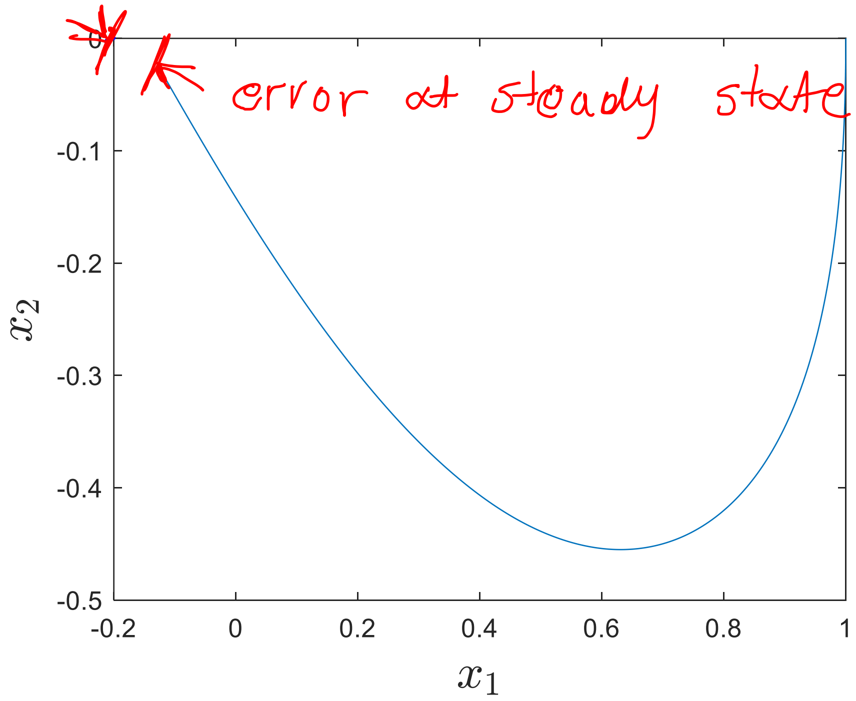
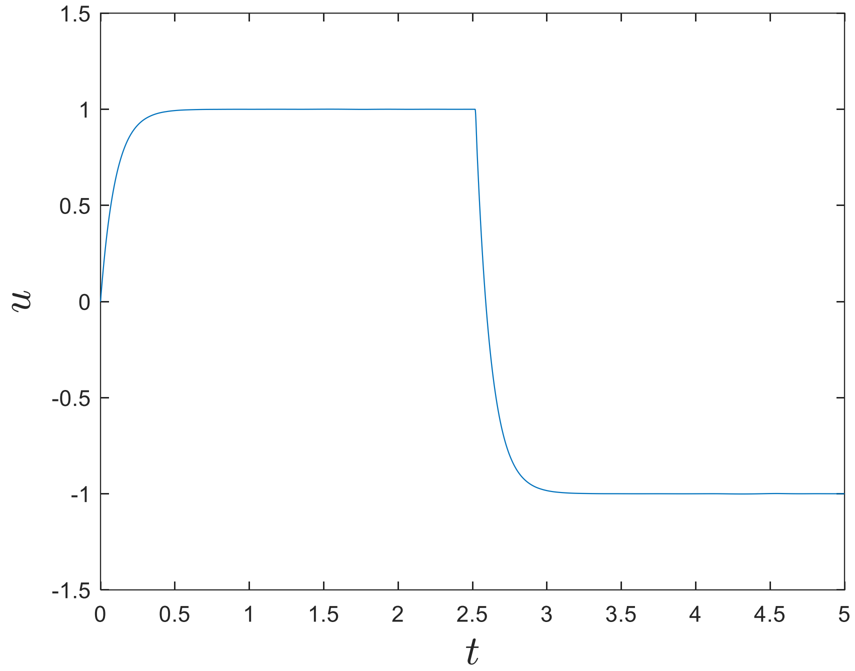
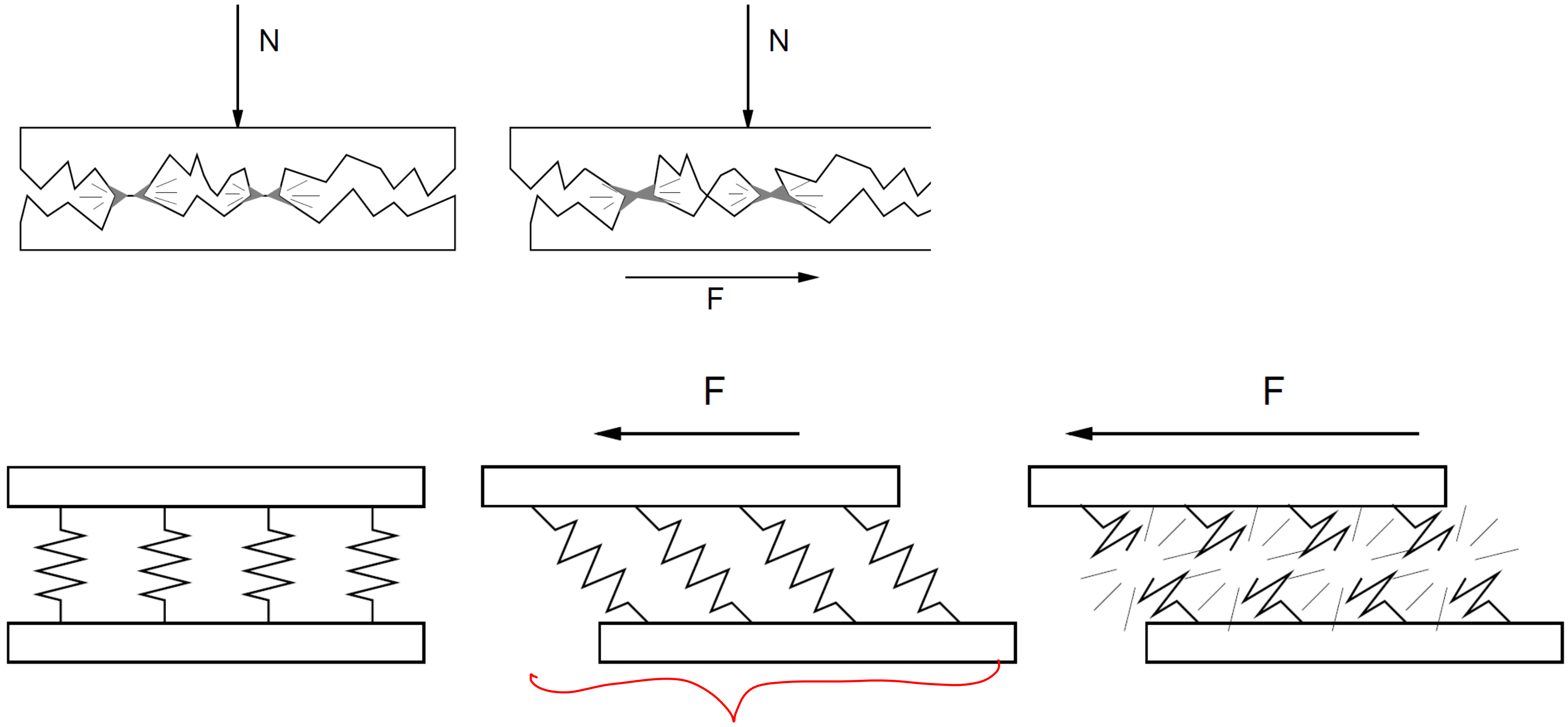
Example

- Continuous control $u = \tanh(x_2 + 2x_1)$

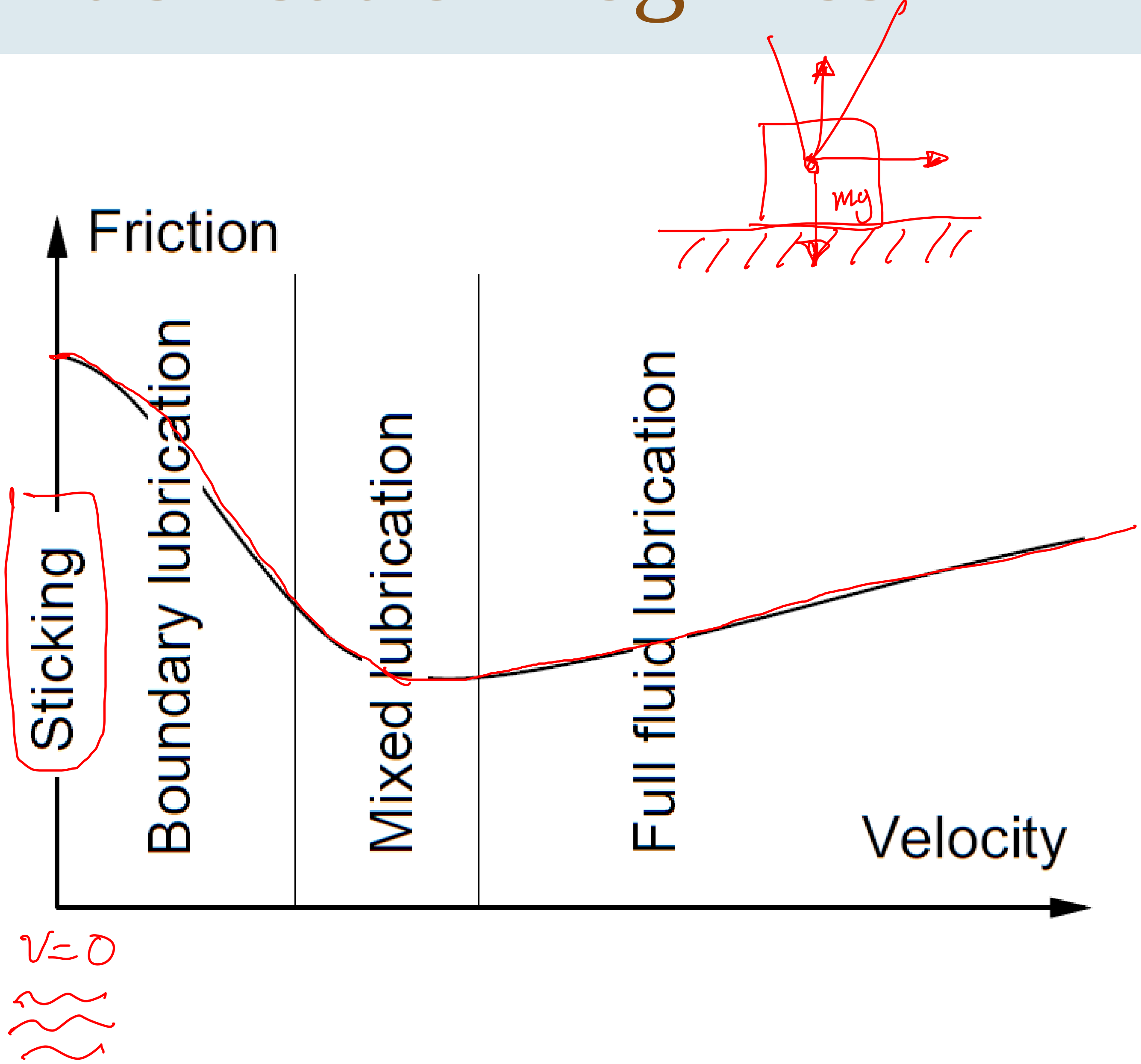


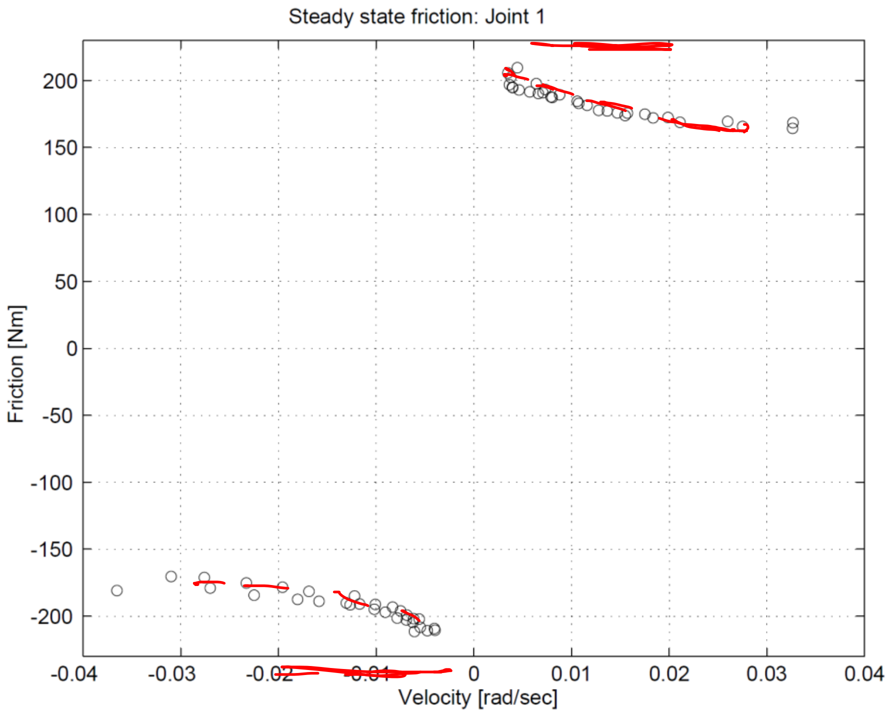


Friction



Lubrication regimes





For low velocity: friction increases with decreasing velocity
Stribeck (1902)

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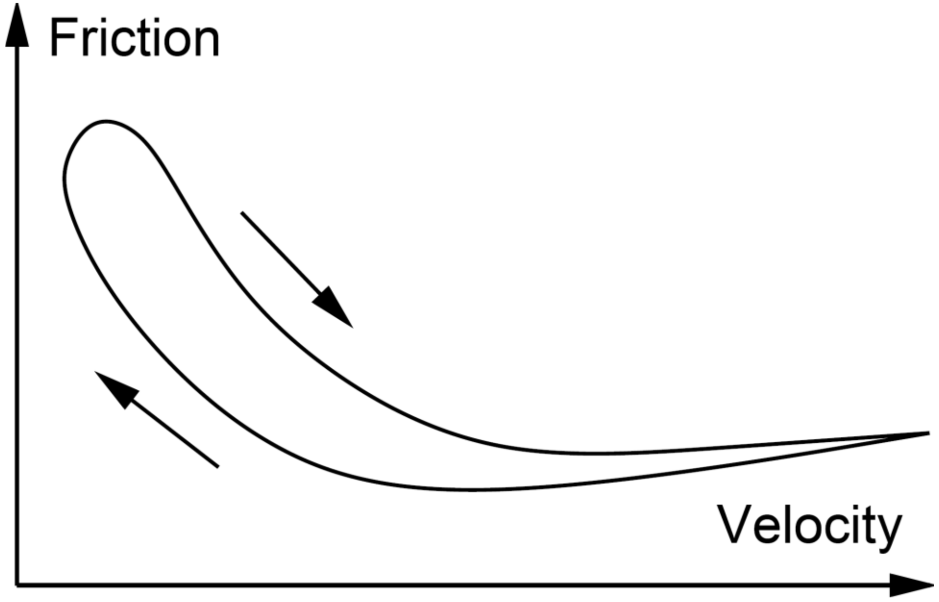
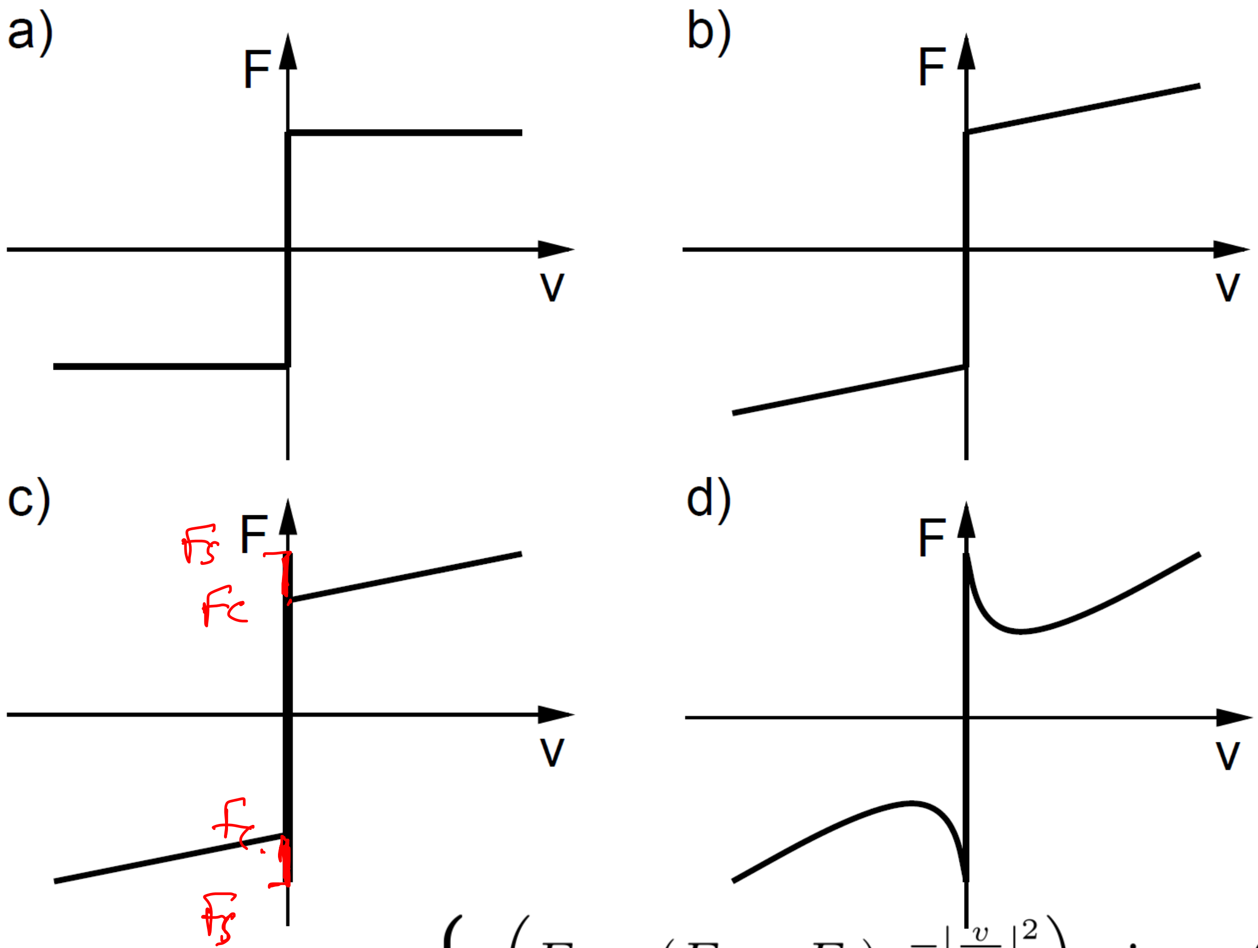
Hysteresis and friction

- Dynamics are important also outside sticking regime
- Hess and Soom (1990)
- Experiment with unidirectional motion $v(t) = v_0 + a\sin(\omega t)$
- Hysteresis effect!



Classical friction models

$m\ddot{x} + \underline{F(t)} = F_e(t)$



$F_e(t) =$ external applied force , F_c, F_v, F_s constants

$$\boxed{F(t) = F_c \text{ sign } v(t)} \quad (a)$$

$$F(t) = \underbrace{F_v v(t)}_{\text{viscous}} + \underbrace{F_c \text{ sign } v(t)}_{\text{Coulomb}} \quad (b)$$

$$F(t) = \begin{cases} F_c \text{ sign } v(t) + F_v v(t) & v(t) \neq 0 \\ \max(\min(F_e(t), F_s), -F_s) & v(t) = 0 \end{cases}$$

(c) equivalent expression

$$F(t) = \begin{cases} \left(F_c + (F_s - F_c)e^{-\left|\frac{v}{v_s}\right|^2}\right) \text{ sign } v(t) + F_v v(t) & v(t) \neq 0 \\ F_e(t) & v(t) = 0 \text{ and } |F_e(t)| < F_s \\ \text{sign}(F_e)F_s & v(t) = 0 \text{ and } |F_e(t)| \geq F_s \end{cases}$$

(d)

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Advanced Friction Models

- Karnopp model
- Armstrong's seven parameter model
- Dahl model
- Bristle model
- Reset integrator model
- Bliman and Sorine
- LuGre model (Lund-Grenoble)

See PhD-thesis by Henrik Olsson

https://lucris.lub.lu.se/ws/portalfiles/portal/4768278/8840259.pdf

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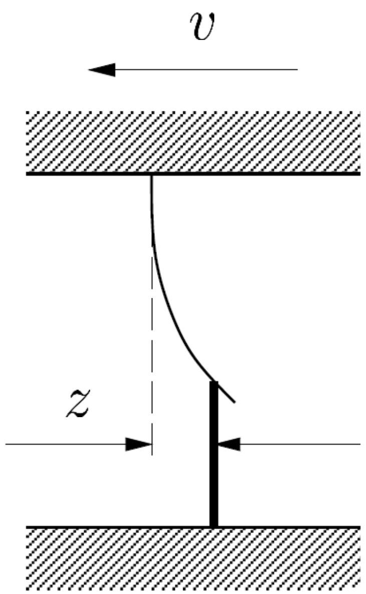
Friction models with extended state

Dahl's model

$$\dot{F}(t) = \sigma_0(v(t) - F_c|v(t)|)$$



LuGre model

$$\dot{z}(t) = v - \frac{|v|}{g(v)}z(t)$$

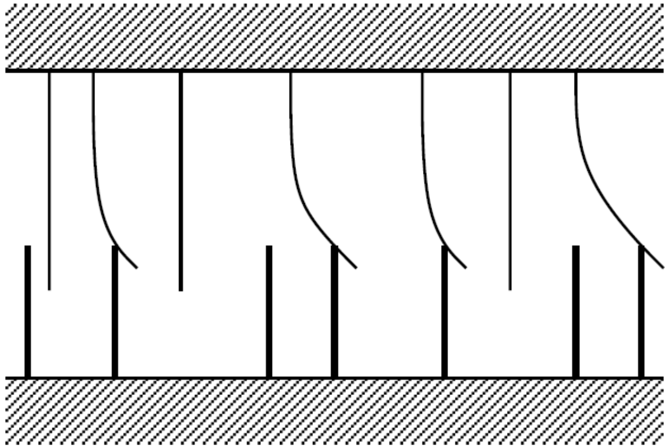
$$g(v) = \left(F_c + (F_s - F_c)e^{-|\frac{v}{v_s}|^2}\right)$$

$$F(t) = \sigma_0 z(t) + \sigma_1 \dot{z} + F_v v(t)$$

stiffness of bristles

damping

viscous friction



Contact between bristles

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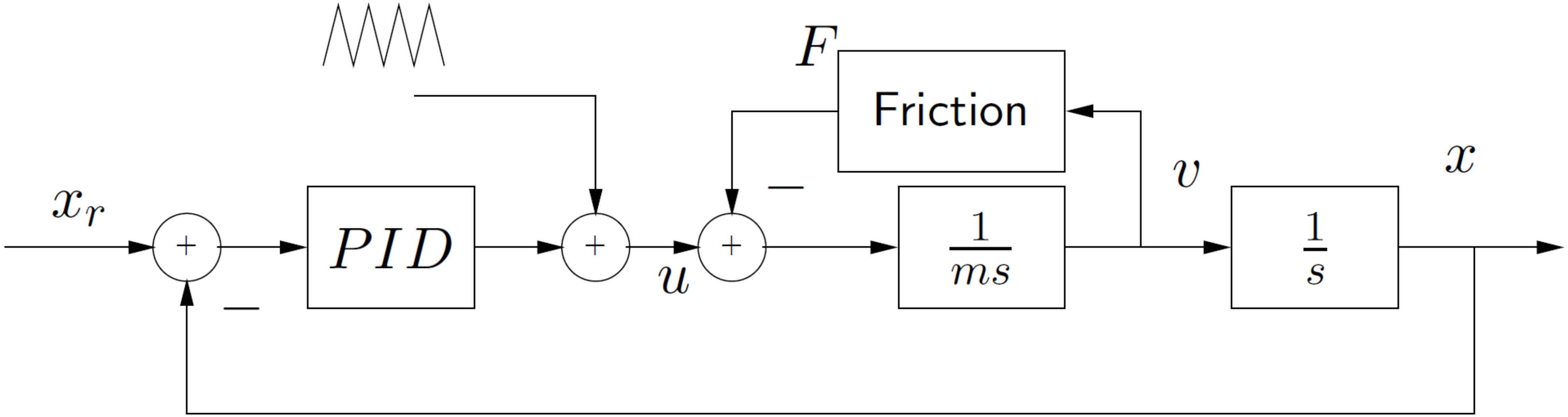
Friction and control

- Friction compensation
  - Lubrication
  - Dither
  - Integral action
  - Non-model based control
  - **Model-based friction compensation**
  - Adaptive friction compensation

High-frequency mechanical vibration: used to avoid sticking

Integral term compensate for slowly varying disturbances



Friction and control

- Friction compensation
  - Lubrication
  - Dither
  - Integral action
  - Non-model based control
  - **Model-based friction compensation**
  - Adaptive friction compensation

- To be useful for control the model should be:
  - sufficiently accurate,
  - suitable for simulation,
  - simple, few parameters to determine.
  - physical interpretations, insight

- Simple models should be preffed.
- If no stiction occurs the v=0-models are not needed.

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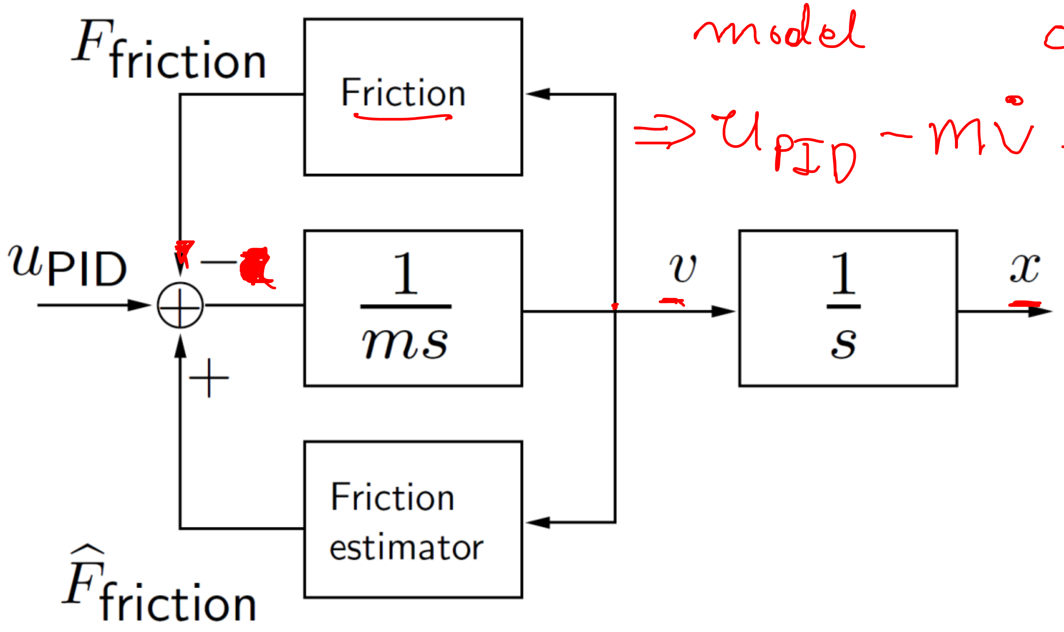
Adaptive friction compensation

$$\boxed{F_{\text{friction}} = a\,\text{sgn}(v)}$$

$\underbrace{m\ddot{x} + F_{\text{friction}}}_{\text{model}} = \underbrace{u_{PID}}_{\text{controller}} + \hat{F}_{\text{friction}}$

$\Rightarrow u_{PID} - m\dot{v} = F - \hat{F}$



Result: $e = a - \hat{a} \to 0$ as $t \to \infty$:

$\frac{d}{dt}\sqrt{v^2} = \text{sgn}(v)\,\dot{v}$

$$\frac{de}{dt} = -\frac{d\hat{a}}{dt} = \frac{dz}{dt} - km\frac{d}{dt}|v|$$
$$= ku_{\text{PID}}\text{sgn}(v) + km\dot{v}\,\text{sgn}(v)$$
$$= k\text{sgn}(v)(u_{\text{PID}} - m\dot{v})$$
$$= -k\text{sgn}(v)(F - \hat{F}) = -k\,\text{sgn}(v)\left(a\cdot\text{sgn}(v) - \hat{a}\,\text{sgn}(v)\right)$$
$$= -k(a - \hat{a})$$
$$= -ke$$

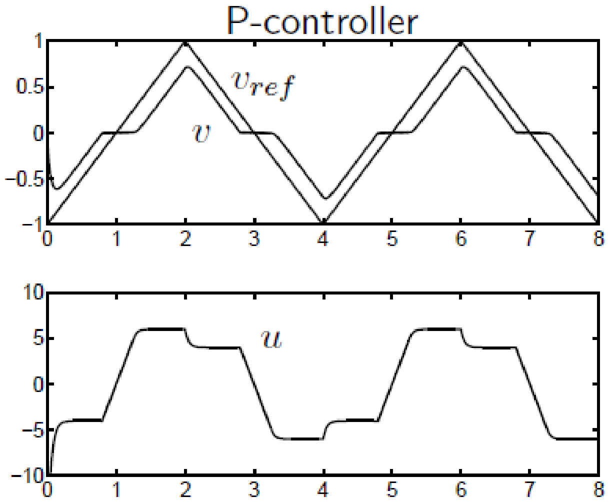
$\Rightarrow$ $\boxed{e(t) = e(0)\,\exp(-kt)}$

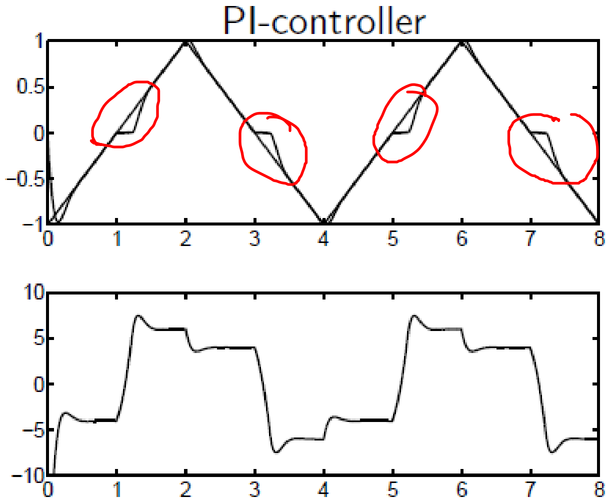
$e(t) \to 0$

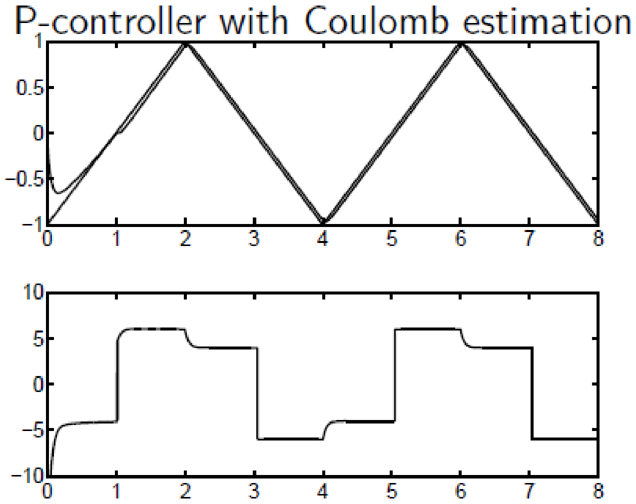
Assumption: v measurable.

Friction estimator:
$$\begin{cases} \dot{z} = ku_{\text{PID}}\text{sgn}(v) & \text{update } e \\ \hat{a} = z - km|v| \\ \widehat{F}_{\text{friction}} = \hat{a}\,\text{sgn}(v) \end{cases}$$

Velocity Control – Results







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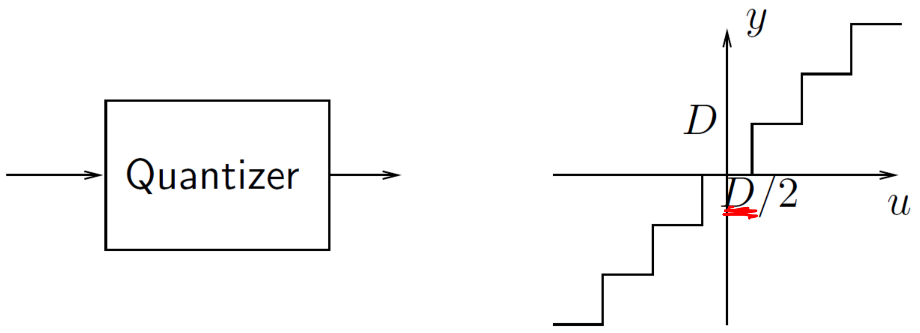
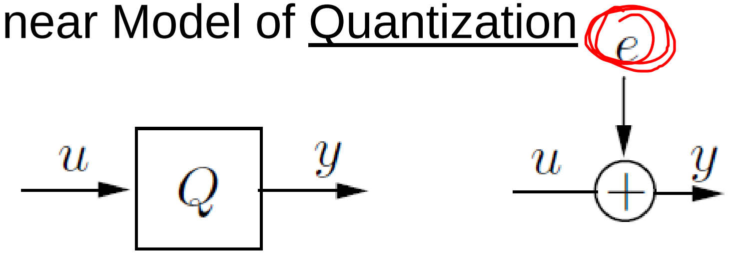
Quantization



- Digital signals have specific number of bits (accuracy and range of signals) (e.g. 8 bits, 64 bits)
- Quantization in A/D and D/A converters
- Quantization of parameters
- Roundoff, overflow, underflow in operations

Linear Model of Quantization



- e: Noise independent of u with rectangular distribution over the quantization size with rectangular distribution with variance

$$Var(e) = \int_{-\infty}^{+\infty} e^2 f_e \, de = \int_{-D/2}^{D/2} e^2 \frac{1}{D} de = \frac{D^2}{12}$$

- It works if quantization level is small compared to the variations in u

For prediction of limit cycles

- Use describing function analysis

Sliding mode control

- Choose K such that $u = v$ drives $\sigma(x)$ to zero in finite time.

$$u = -K\text{sign}(\sigma)$$

$$\text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$$

Sliding mode control

- Choose K such that $u = v$ drives $\sigma(x)$ to zero in finite time.

$$u = -K\text{sign}(\sigma)$$

$$\text{sign}(\sigma) = \begin{cases} 1, & \sigma > 1 \\ -1, & \sigma < 1 \end{cases}$$

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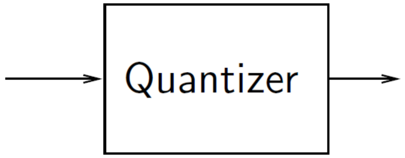
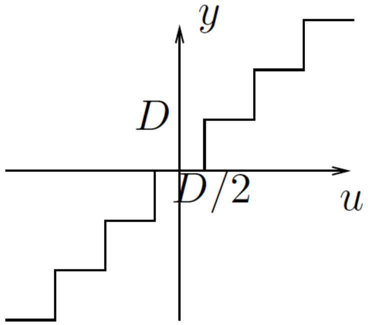
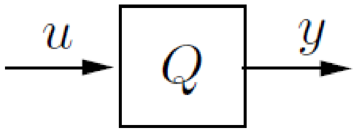
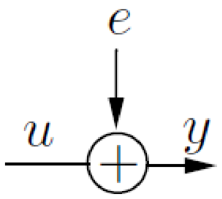
Quantization: Describing function

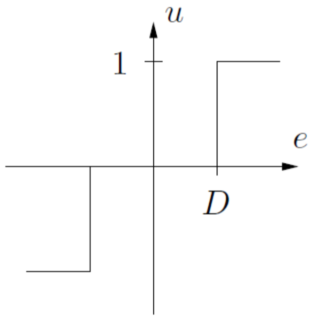


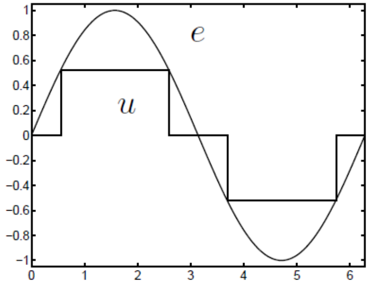






- Recall the deadzone nonlinearity





$$Var(e) = \int_{-\infty}^{+\infty} e^2 f_e \, de = \int_{-D/2}^{D/2} e^2 \frac{1}{D} de = \frac{D^2}{12}$$