FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 12: From optimal control to nonlinear control

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Outline

Linear Quadratic Control

Time Optimal Control

Model Predictive Control

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Maximum principle – no final time constraint

Optimization Problem (OP2)

Minimize $\displaystyle\int_0^{t_f} \overbrace{L(x(t),u(t))}^{\text{Trajectory cost}}\,dt + \overbrace{\phi(t_f,x(t_f))}^{\text{Final cost}}$

where

$x(t) \in R^n, \quad u(t) \in U \subseteq R^m, \quad 0 \le t \le t_f$

$\dot{x}(t) = f(x(t),u(t)), \qquad x(0) = x_0$

$\psi(t_f,x(t_f)) = 0$

Assume that OP2 has a solution. Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$ and a vector $\mu \in R^r$ such that $[n_0 \;\; \mu^T] \neq 0$ and

$$H(x,u,\lambda) = n_0 L(x,u) + \lambda^T(t) f(x,u)$$

$$\min_{u \in U} H(x^*(t),u,\lambda(t),n_0) = H(x^*(t),u^*(t),\lambda(t),n_0), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$\begin{cases} \dot{\lambda}(t) & = -H_x^T(x^*(t),u^*(t),\lambda(t),n_0) \\ \lambda(t_f) & = n_0\phi_x^T(t_f,x^*(t_f)) + \psi_x^T(t_f,x^*(t_f))\mu \end{cases}$$

If the end time t_f is free, then $H(x^*(t_f),u^*(t_f),\lambda(t_f),n_0) = 0$.
If the end time t_f is given:
$H(x^*(t_f),u^*(t_f),\lambda(t_f),n_0) = -n_0\phi_t(t_f,x^*(t_f)) - \mu^T\psi_t(t_f,x^*(t_f))$.

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Optimal control (Linear Control Systems with quadratic running cost)

Performance, cost function

Free-final-state but in the perfromance function

$$J(u) = \frac{1}{2}x^T(t_f)P(t_f)x(t_f) + \frac{1}{2}\int_0^{t_f} x^TQx + u^TRu\,dt$$

System dynamics
(dynamic constraint)

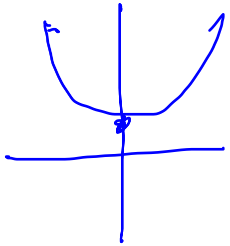
$$\dot{x} = f(x,u),\ x(t_0) = x_0$$

Hamiltonian

$$H(x,u,\lambda) = \tfrac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu) = u^TA_1u + b^Tu + c$$

Hamiltonian minimization with respect to u

$$\frac{\partial H(u)}{\partial u} = u^TR + \lambda^TB \rightarrow \boxed{u = -R^{-1}B^T\cancel{\lambda}P}\,x$$



State equation

Co-state, adjoint equation

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix} \qquad \begin{matrix} x(t_0) = x_0 \\ \lambda(t_f) = P(t_f)x(t_f)\end{matrix}$$

$$\lambda(t) = P(t)x(t)$$

$$\begin{cases}\dot{x}(t) = (A(t) - BR^{-1}B^TP(t))x(t), \quad x(0) \text{ given}\\ \dot{P}(t) = P(t)A + A^TP(t) + P(t)BR^{-1}B^T(t)P(t) - Q\end{cases}$$

$P(t_f)$ known by the cost function

Optimal control (Linear Control Systems with quadratic running cost and fixed final state)

Performance, cost function $$J(u) = \frac{1}{2}\int_0^{t_f} u^T R u \, dt$$

Constraint on $$\psi(x(t_f)) = 0 \rightarrow x(t_f) = r$$

System dynamics (dynamic constraint) $$\dot{x} = f(x,u),\ x(t_0) = x_0$$

Hamiltonian $$H(x,u,\lambda) = \frac{1}{2}u^T R u + \lambda^T (Ax + Bu)$$

Hamiltonian minimization with respect to u

$$\frac{\partial H(u)}{\partial u} = u^T R + \lambda^T B \rightarrow u = -R^{-1}B^T\lambda$$

State equation

Co-state, adjoint equation

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ O & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad \begin{matrix} x(t_0) = x_0 \\ \lambda(t_f)\ ?? \end{matrix}$$

$\dot{x} = -\alpha x + f$

Steps: 1. solve backwards the second differential equation $\lambda(t) = e^{A^T(t_f - t)}\lambda(t_f)$

2. substitute the solution in the system dynamics and solve the initial value problem

3. use the constraint to find $\lambda(t_f)$

$x(t_f) = r$

$u^*(t)$

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Example – moving to the origin in minimum time

$$\min \int_0^{t_f} dt$$

min t_f

$$\text{s.t.} \quad \ddot{x} = u, \quad x(0) = x_0,\ \dot{x}(0) = v_0$$

$$u \in [-1, 1]$$

$$x(t_f) = 0, \quad \dot{x}(t_f) = 0$$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

Hamiltonian: $H = n_0 + \lambda_1 x_2 + \lambda_2 u$

Adjoint equation:

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = -H_x^T = \begin{bmatrix} -\frac{\partial H}{\partial x_1} \\ -\frac{\partial H}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix} \qquad \begin{cases} \lambda_1(t_f) = \mu_1 \\ \lambda_2(t_f) = \mu_2 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1(t) = \mu_1 \\ \lambda_2(t) = \mu_1(t - t_f) + \mu_2 \end{cases}$$

At optimality: $\min_u H(u) \equiv \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\sigma(t) = \lambda_2(t)} u + n_0 + \lambda_1 x_2$

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$

Conditions: $H(t_f) = 0 \Rightarrow \mu_2 u^*(t_f) = -n_0$

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Example – minimum time control

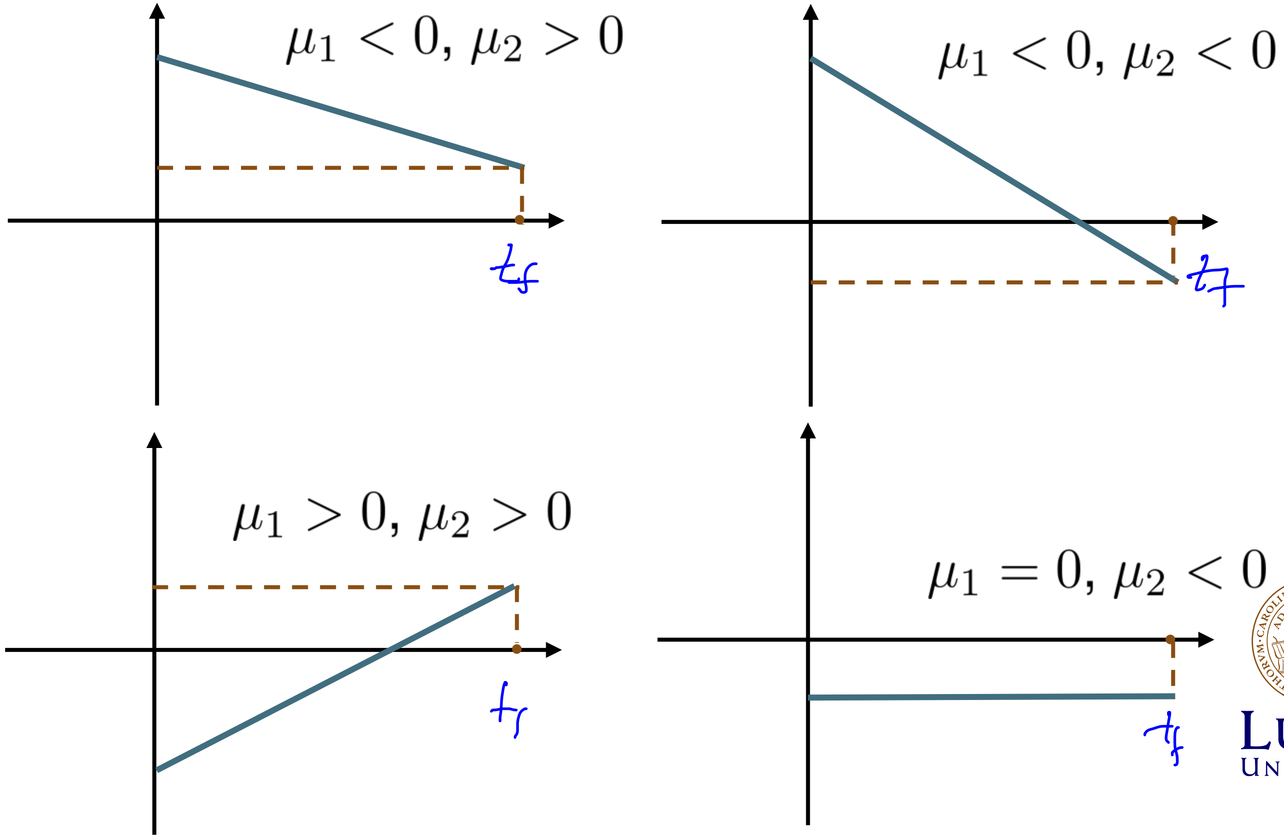
$$\min \int_0^{t_f} dt$$

$$\text{s.t.} \quad \ddot{x} = u, \quad x(0) = x_0, \; \dot{x}(0) = v_0$$
$$u \in [-1, 1]$$
$$x(t_f) = 0, \quad \dot{x}(t_f) = 0$$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

$$H = (\mu_1(t - t_f) + \mu_2)u + n_0 + \lambda_1 x_2$$
$$\mu_2 u^*(t_f) = -n_0$$
$$\sigma(t) = \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\lambda_2(t)}$$

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$



Example – minimum time control

$$\min \int_0^{t_f} dt$$

$$H = (\mu_1(t - t_f) + \mu_2)u + n_0 + \lambda_1 x_2$$

$$\mu_2 u^*(t_f) = -n_0$$

$$\sigma(t) = \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\lambda_2(t)} \qquad u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$

$\lambda_2(t)$ ⇝ monotonically decreasing / increasing function of time

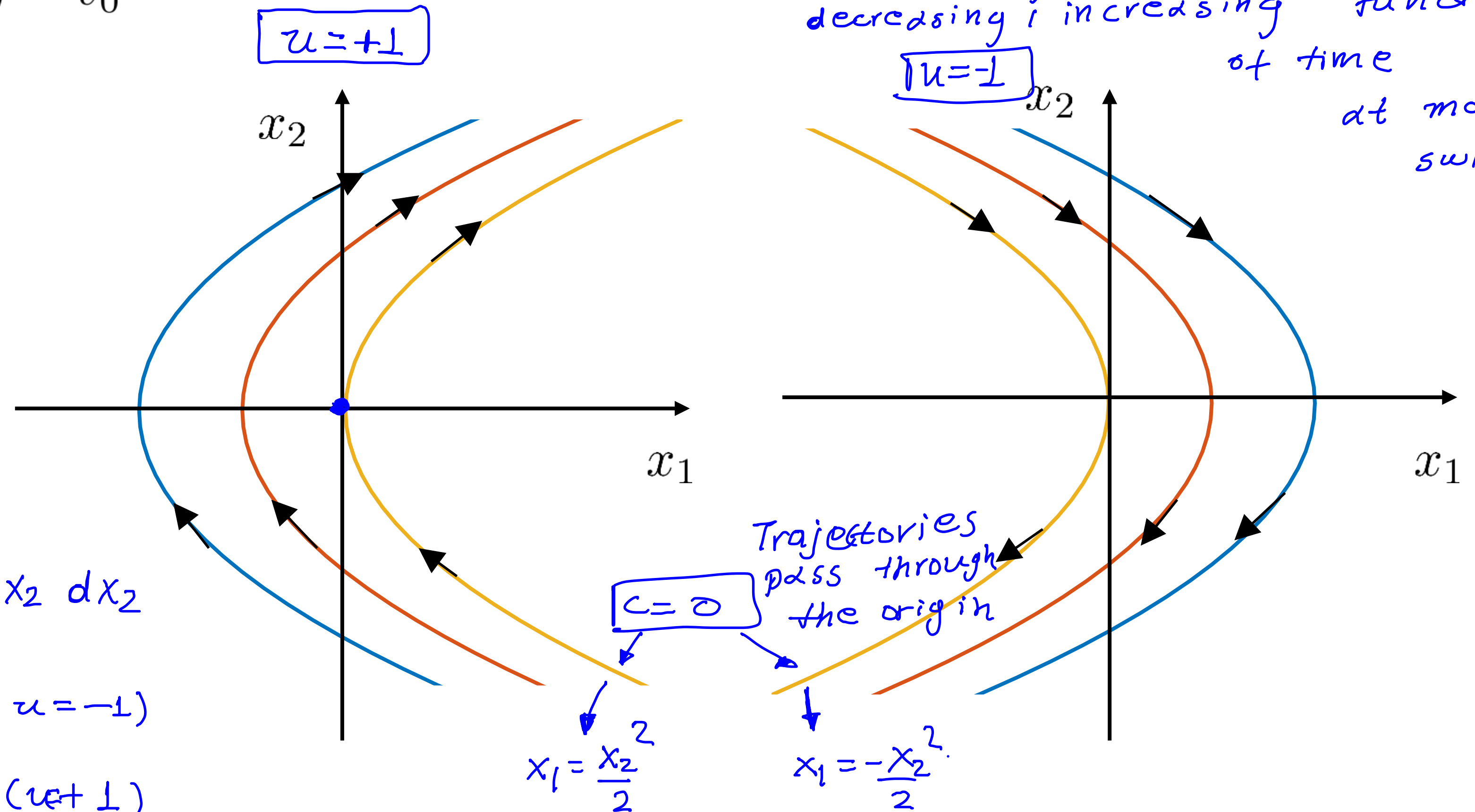
at most one switching

s.t. $\ddot{x} = u, \quad x(0) = x_0,\ \dot{x}(0) = v_0$

$u \in [-1, 1]$

$x(t_f) = 0, \quad \dot{x}(t_f) = 0$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$



$$\Rightarrow \frac{dx_1}{dx_2} = \frac{x_2}{u} \Rightarrow u\,dx_1 = x_2\,dx_2$$

$$\Rightarrow \begin{cases} x_1 = -\frac{x_2^2}{2} + c, & \sigma > 0 \ (u = -1) \\ x_1 = \frac{x_2^2}{2} + c, & \sigma < 0 \ (u = +1) \end{cases}$$

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Example – minimum time control

See Ex. 7.11

$$\min \int_0^{t_f} dt$$

$$H = (\mu_1(t - t_f) + \mu_2)u + n_0 + \lambda_1 x_2$$

$$\mu_2 u^*(t_f) = -n_0$$

$$\sigma(t) = \underbrace{(\mu_1(t - t_f) + \mu_2)}_{\lambda_2(t)}$$

$$u^*(t) = \begin{cases} -1, & \sigma(t) > 0 \\ 1, & \sigma(t) < 0 \end{cases}$$

Can be seen as feedback control

s.t. $\ddot{x} = u, \quad x(0) = x_0, \; \dot{x}(0) = v_0$

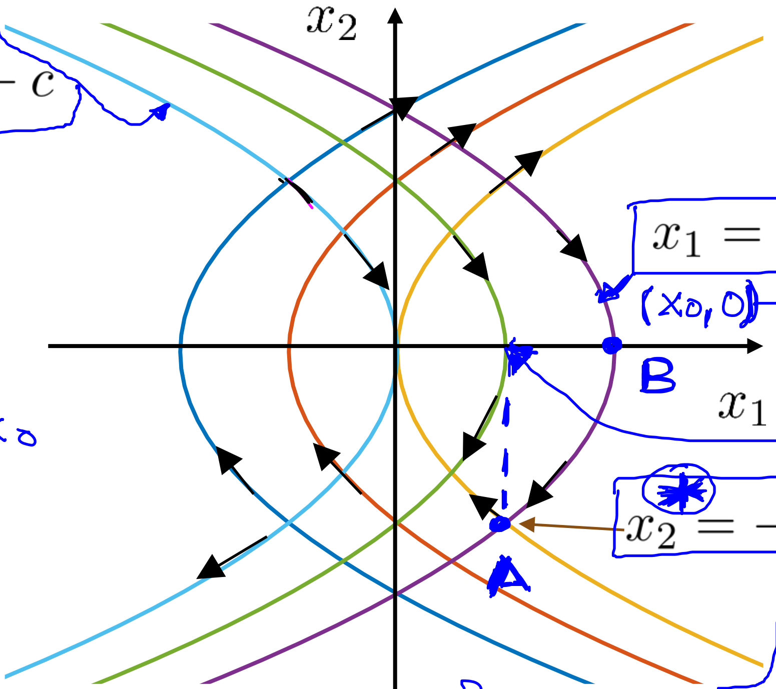
$u \in [-1, 1]$

$x(t_f) = 0, \quad \dot{x}(t_f) = 0$

$$u(t) = -\operatorname{sgn}\left[x_1(t) + \frac{1}{2}\operatorname{sgn}(x_2)\, x_2^2(t)\right]$$

$\boxed{c = 0 \quad x_1 = -\frac{1}{2}x_2^2 + c}$

$u = -1$



$\boxed{x_1 = -\frac{1}{2}x_2^2 + x_0}$ $(x_0, 0)$

$v_0 = 0, \; x_0 > 0$

$\boxed{x_1 = \frac{x_0}{2}}$

$\boxed{x_2 = -\sqrt{x_0}}$

Time from B → A $\quad t_{AB} = \sqrt{x_0}$

$\boxed{x_1 = \frac{1}{2}x_2^2 + c \quad c = 0}$

$$\ddot{x} = u \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

(*) Intersection of $\begin{cases} x_1 = -\frac{1}{2}x_2^2 + x_0 \\ x_1 = \frac{1}{2}x_2^2 \end{cases}$

$A\left(\frac{x_0}{2}, -\sqrt{x_0}\right)$

Time B → A, $u = -1$ $\quad \dot{x}_2 = u$

$(v_A - v_B) = -1 \cdot t_{AB} \;\rightarrow\; t_{AB} = \sqrt{x_0}$

Time A → 0

$(v_0 - v_A) = 1 \cdot t_{0A} \;\rightarrow\; t_{0A} = \sqrt{x_0}$

$\Rightarrow t = 2\sqrt{x_0}$

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- Optimal control provides with trajectories that can be then used as references to a controller
- We mainly address input constraints, but there might be state constraints e.g. obstacles that need to be avoided
- A popular approach is Model Predictive Control

From Continuous to Discrete Time

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Sampling time T

$$A_d = e^{AT}$$
$$B_d = (A_d - I)A^{-1}B$$

Euler Approximation

$$A_d = I + TA$$
$$B_d = TB$$

$$x(t+1) = A_d x(t) + B_d u(t)$$
$$y(t) = Cx(t)$$

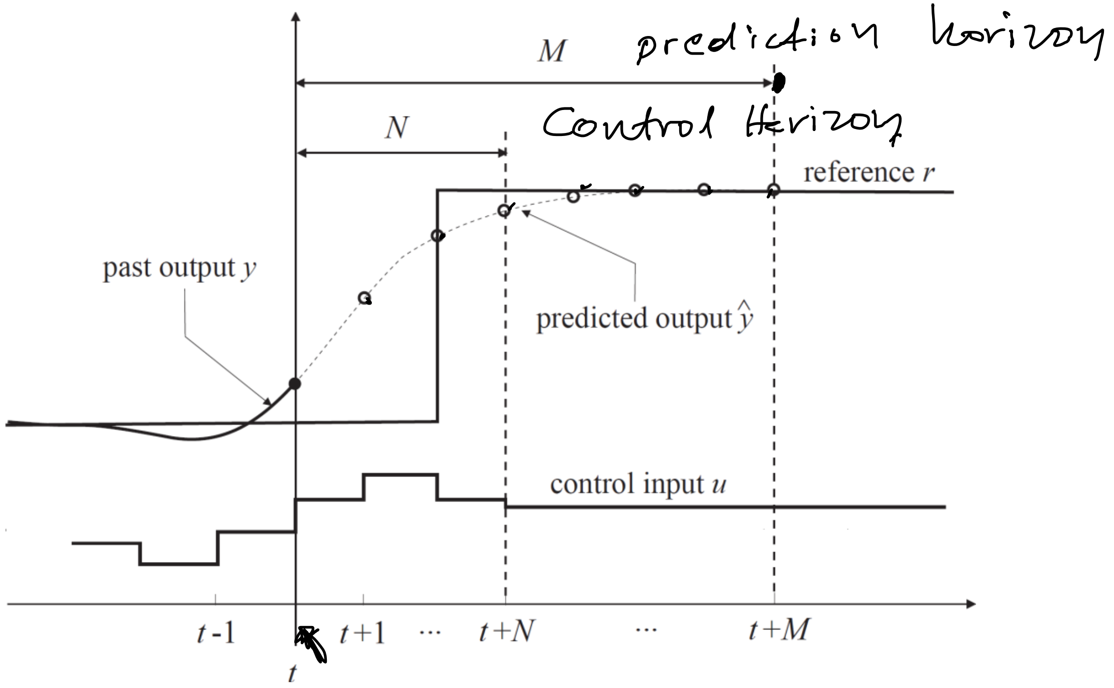
$$J = \frac{1}{2}x^T(t_f)P(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x^TQx + u^TRu\,dt$$

The integral becomes sum

$$J = x^T(M)P(M)x(M) + \sum_{1}^{M-1} x^T(i)Qx(i) + u^T(i)Ru(i)$$

$$J = y^T(M)P(M)y(M) + \sum_{1}^{M-1} y^T(i)Qy(i) + u^T(i)Ru(i)$$

Receding horizon control – basic idea



1. At time instant t predict the response of the system over a prediction horizon M using inputs over a control horizon N.

2. Optimize a specified objective or cost function with respect to a control sequence $u(t+j), \quad j = 0, 1, \ldots, N-1$.

3. Apply the first control $u(t)$ and start over from 1 at next sample.

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Unroll the cost

Minimize a cost function, V, of inputs and predicted outputs.

$$V = V(U_t, Y_t), \qquad U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \qquad Y_t = \begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix}$$

V often quadratic

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t \tag{1}$$

$\Longrightarrow$ linear controller

$$u(t) = -L\widehat{x}(t|t)$$

Optimization problem

$$J(t) = y^T(t+M)P(t+M)y(t+M) + \sum_{t+1}^{t+M-1} y^T(i)Qy(i) + u^T(i)Ru(i)\,dt$$

Minimize a cost function, V, of inputs and predicted outputs.

$$V = V(U_t, Y_t), \qquad U_t = \begin{bmatrix} u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}, \qquad Y_t = \begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix}$$

V often quadratic

$$V(U_t, Y_t) = Y_t^T Q_y Y_t + U_t^T Q_u U_t$$

$\Longrightarrow$ linear controller

$$u(t) = -L\widehat{x}(t|t)$$

Prediction

Discrete-time model

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + B_v v_1(t) \\
y(t) &= Cx(t) + v_2(t)
\end{aligned}
\qquad t = 0, 1, \ldots
$$

Predictor (v unknown)

$$
\begin{aligned}
\widehat{x}(t+k+1|t) &= A\widehat{x}(t+k|t) + Bu(t+k) \\
\widehat{y}(t+k|t) &= C\widehat{x}(t+k|t)
\end{aligned}
$$

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Receding horizon control

- $\widehat{x}(t|t)$ is predicted by a standard Kalman filter, using outputs up to time t, and inputs up to time $t-1$.
- Future predicted outputs are given by

$$\begin{bmatrix} \widehat{y}(t+M|t) \\ \vdots \\ \widehat{y}(t+1|t) \end{bmatrix} = \begin{bmatrix} CA^M \\ \vdots \\ CA \end{bmatrix} \widehat{x}(t|t) + \begin{bmatrix} CB & CAB & CA^2B & \dots \\ 0 & CB & CAB & \dots \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} u(t+M-1) \\ \vdots \\ u(t+N-1) \\ \vdots \\ u(t) \end{bmatrix}$$

$$Y_t = D_x \widehat{x}(t|t) + D_u U_t$$

Receding horizon control

- + Flexible method
  - ⋆ Many types of models for prediction:
    - ⋆ state space, input–output, step response, nonlinear models
  - ⋆ MIMO
  - ⋆ Time delays
- + Can include constraints on input signal and states
- + Can include future reference and disturbance information
- – On-line optimization needed
- – Stability (and performance) analysis can be complicated

Receding horizon control

Limitations on control signals, states and outputs,

$$|u(t)| \leq C_u \quad |x_i(t)| \leq C_{x_i} \quad |y(t)| \leq C_y,$$

leads to linear programming or quadratic optimization.

Efficient optimization software exists.

Design of MPC

- Model
- M (look on settling time)
- N as long as computational time allows
- If $N < M - 1$ assumption on $u(t+N), \ldots, u(t+M-1)$ needed (e.g., $= 0$, $= u(t+N-1)$.)
- Q_y, Q_u (trade-offs between control effort and performance)
- C_y, C_u constraints often given
- Sampling time