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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 11: Optimal control 2

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Example 2

$$\min \int_0^1 u^4 dt + x(1)$$

$$\dot{x} = -x + u, \quad x(0) = 0$$

$$\min \int_0^{t_f} \overbrace{L(x(t), u(t))}^{\text{Trajectory cost}} dt + \overbrace{\phi(x(t_f))}^{\text{Final cost}}$$

where

$x(t) \in R^n$

$u(t) \in U \subseteq R^m$

$\dot{x}(t) = f(x(t), u(t))$

$x(0) = 0$

Hamiltonian:

$$H = L + \lambda^T \cdot f = u^4 + \lambda(-x + u)$$

Adjoint equation:

Final value problem

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -(-\lambda) \implies \lambda(t) = Ce^t$$

$$\lambda(t_f) = \frac{\partial \phi}{\partial x} = 1 \quad \lambda(1) = 1 = Ce^1 \implies \lambda(t) = e^{t-1}$$

$\Rightarrow C = e^{-1}$

Optimality: $H_u = .. \frac{\partial H}{\partial u} = 4u^3 + \lambda$

$H_u = 0 \Rightarrow u = \sqrt[3]{-\frac{\lambda}{4}}$

Maximum principle

Optimization Problem (OP2)

$$\text{Minimize} \int_0^{t_f} \overbrace{L(x(t), u(t))}^{\text{Trajectory cost}}\, dt + \overbrace{\phi(t_f, x(t_f))}^{\text{Final cost}}$$

where

$$x(t) \in R^n, \quad u(t) \in U \subseteq R^m, \quad 0 \le t \le t_f$$

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0$$

$$\psi(t_f, x(t_f)) = 0$$

Assume that OP2 has a solution. Then there is a vector function $\lambda(t)$, a number $n_0 \ge 0$ and a vector $\mu \in R^r$ such that $[n_0 \;\; \mu^T] \ne 0$ and

$$H(x, u, \lambda) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

$$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \le t \le t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$\begin{cases} \dot{\lambda}(t) & = -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\ \lambda(t_f) & = n_0 \phi_x^T(t_f, x^*(t_f)) + \psi_x^T(t_f, x^*(t_f))\mu \end{cases}$$

If the end time t_f is given, then $H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0.$
If the end time t_f is free:
$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = -n_0 \phi_t(t_f, x^*(t_f)) - \mu^T \psi_t(t_f, x^*(t_f)).$

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Remarks

- Can scale $n_0, \mu, \lambda(t)$ by the same constant

$$\begin{cases} \dot{\lambda}(t) & = -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\ \lambda(t_f) & = n_0 \phi_x^T(t_f, x^*(t_f)) + \psi_x^T(t_f, x^*(t_f))\mu \end{cases}$$

$$H(x, u, \lambda) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

- Can reduce to two cases

$$H(x, u, \lambda) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

  - $n_0 = 1$ (normal)

If $n_0 > 0$, renormalize to $n_0 = 1$. Only existence of positive n_0 matters.

  - $n_0 = 0$ (abnormal, since L and ϕ don't matter)

- Fixed time t_f and no end constraints $\Rightarrow$ normal case

Optimal control (Linear Control Systems with quadratic running cost)

Performance, cost function

$$J(u) = \frac{1}{2}x^T(t_f)P(t_f)x(t_f) + \frac{1}{2}\int_0^{t_f} x^TQx + u^TRu\,dt$$

System dynamics
(dynamic constraint)

$$\dot{x} = f(x,u),\; x(t_0) = x_0$$

Hamiltonian

$$H(x,u,\lambda) = \tfrac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu)$$

Hamiltonian minimization with respect to u

$$\frac{\partial H(u)}{\partial u} = u^TR + \lambda^TB \rightarrow u = -R^{-1}B^T\lambda \qquad \lambda(t) = P(t)x(t)$$

State equation

Co-state, adjoint equation

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix} \qquad \begin{matrix}x(t_0) = x_0\\ \lambda(t_f) = P(t_f)x(t_f)\end{matrix}$$

$$\dot{P}(t) = P(t)A + A^TP(t) + P(t)BR^{-1}B^T(t)P(t) - Q$$

$P(t_f)$ known by the cost function

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Example – optimal heating (minimum fuel problem)

$$\min \int_0^{t_f=1} P(t)\,dt$$

$$\text{s.t.} \quad \dot{T} = P - T, \quad T(0) = 0$$
$$0 \le P \le P_{max}$$
$$T(1) = 1$$

T temperature
P heat effect

Hamiltonian

$$H = n_0 P + \lambda(P - T)$$

Adjoint equation

$$\dot{\lambda}^T = -H_T = -\frac{\partial H}{\partial T} = \lambda^T \qquad \lambda(1) = \mu$$

$$\Rightarrow \quad \lambda(t) = \mu e^{t-1}$$
$$\Rightarrow \quad H = \underbrace{\left(n_0 + \mu e^{t-1}\right)}_{\sigma(t)} P - \lambda T$$

H cannot be minimized by $H_u = 0$
The constraint on P makes the problem well-defined

At optimality

$$P^*(t) = \begin{cases} 0, & \sigma(t) > 0 \\ P_{max}, & \sigma(t) < 0 \end{cases}$$

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Example – optimal heating

$$\min \int_0^{t_f=1} P(t)\,dt$$

$$\text{s.t.}\quad \dot T = P - T, \quad T(0) = 0$$
$$0 \le P \le P_{max}$$
$$T(1) = 1$$

$$H = \sigma(t)P - \lambda T$$

$$\sigma(t) = n_0 + \underbrace{\mu e^{t-1}}_{\lambda}$$

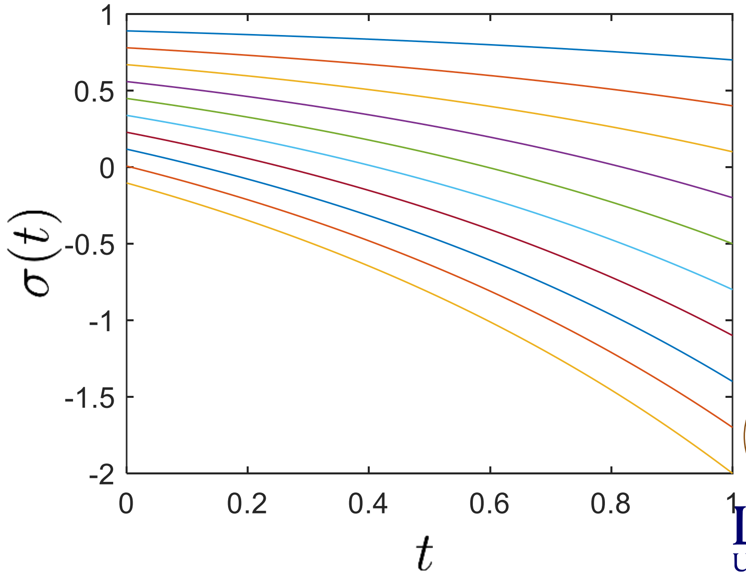
$$P^*(t) = \begin{cases} 0, & \sigma(t) > 0 \\ P_{\max}, & \sigma(t) < 0 \end{cases}$$

$$\left.\begin{array}{l} \mu > 0 \Rightarrow \sigma(t) > 0 \text{ for all } t \\ \\ \mu = 0 \Rightarrow \lambda(t) = 0\ \forall t \Rightarrow \sigma(t) = n_0 > 0\ \forall t \end{array}\right\} \begin{array}{l} \Rightarrow P \equiv 0 \Rightarrow \dot T = -T,\ T(0) = 0 \\ \Rightarrow T(1) = 0 \neq 1 \end{array}$$

$$\mu < 0 \Rightarrow \sigma(t) \text{ strictly decreasing for all } t$$

$$\Rightarrow \dot T = -(T - P_{\max}),\ T(t_1) = 0$$
$$\Rightarrow T(t) = \left[1 - e^{-(t-t_1)}\right] P_{\max}$$

T temperature
P heat effect



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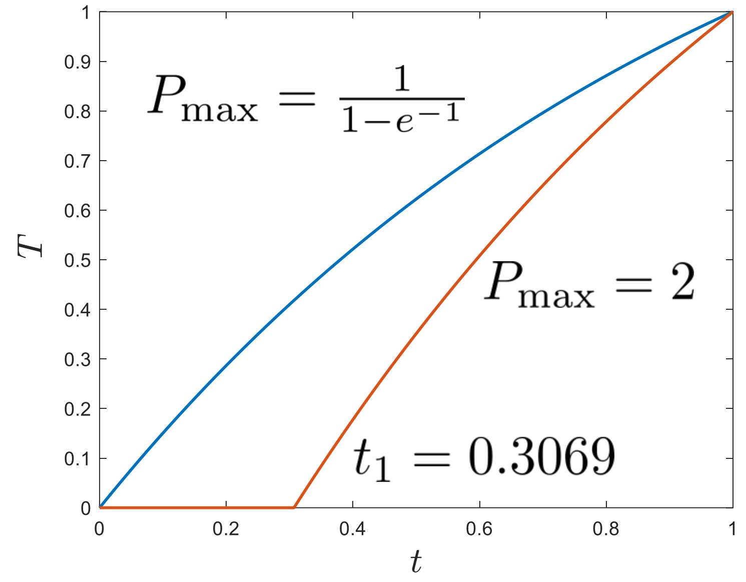
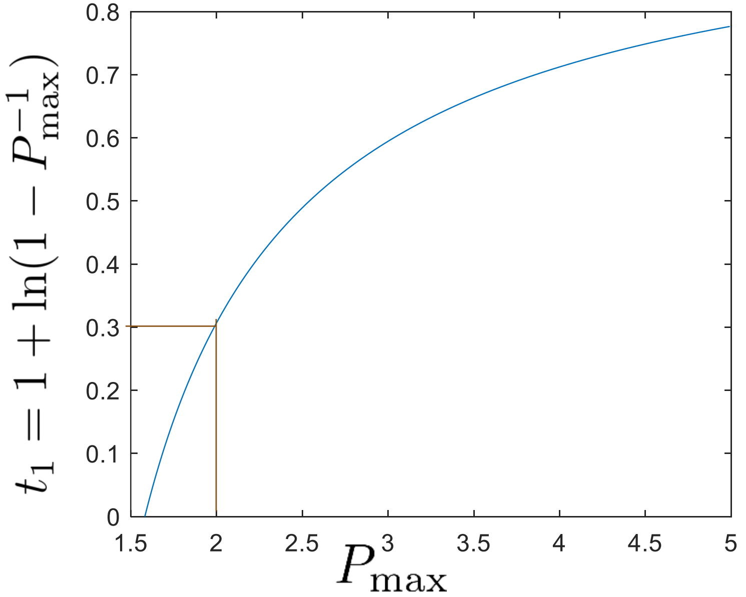
Example – optimal heating





If $t_1 = 0$ (no switching)

$$P_{\max} = \frac{1}{1-e^{-1}}$$

If $0 < t_1 < 1$ (switching from 0 to $P_{\max}$)

$$T(1) = \left[1 - e^{-(1-t_1)}\right] P_{\max} = 1$$

$$\Downarrow$$

$$t_1 = 1 + \ln(1 - P_{\max}^{-1})$$

$$P_{\max} > \frac{1}{1-e^{-1}}$$

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