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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 10: Optimal control I

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Simplified Adaptive control

$$\begin{cases} \dot{x} &= \theta x^2 + u \\ u &= -\hat{\theta}(t) x^2 + v \end{cases} \Rightarrow \dot{x} = \tilde{\theta} x^2 + v$$

Design:

- an update law for $\hat{\theta}$, $\dot{\hat{\theta}} = \ldots$.
- a control signal $v(x)$ $= -kx$

such that $x \to 0$

Introduce the new state $\tilde{\theta} = \theta - \hat{\theta}$.
Find $\dot{x} = f(x, \tilde{\theta}, v)$

$\dot{\tilde{\theta}} = \dot{\theta} - \dot{\hat{\theta}}$, $\dot{\theta} = 0$

$\boxed{\dot{\tilde{\theta}} = -w}$

Let us try the Lyapunov function $\begin{cases} V = \frac{1}{2}(x^2 + \gamma \tilde{\theta}^2) \\ \dot{V} = x \cdot \dot{x} + \gamma \tilde{\theta} \dot{\tilde{\theta}} = x(\tilde{\theta} x^2 + v) - \gamma \tilde{\theta} w \end{cases}$

$= \tilde{\theta} x^3 - \gamma w \tilde{\theta} + x v$ ← $v = -kx$

What do we prove if $\dot{V} \leq 0$?

$w = \frac{1}{\gamma} x^3$

$= -kx^2$

$\dot{V} \leq 0$ → Lasalle $M = \{\tilde{\theta} \in \mathbb{R}, x = 0\}$

Set $\hat{\theta}(t) = \theta$

What principle of design is used?

$\rightarrow \begin{rcases} \dot{x} = \theta x^2 + u \\ u = -\theta x^2 + v \\ v = -kx, \; k > 0 \end{rcases} \Rightarrow$ such that $x \to 0$ where θ is a constant.

$\dot{x} = -kx \Rightarrow x \to 0$ exponentially

$$\begin{cases} \dot{x} = \tilde{\theta} x^2 + v \\ \dot{\tilde{\theta}} = -w \end{cases}$$

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Outline

- Static optimization
- Problem formulation
- Maximum principle
- Examples

Optimal Control

Idea: Formulate the design problem as optimization problem

+ Gives systematic design procedure
+ Can use on nonlinear models
+ Can capture limitations etc as constraints
– Hard to find suitable criterium?!
– Can be hard to find the optimal controller

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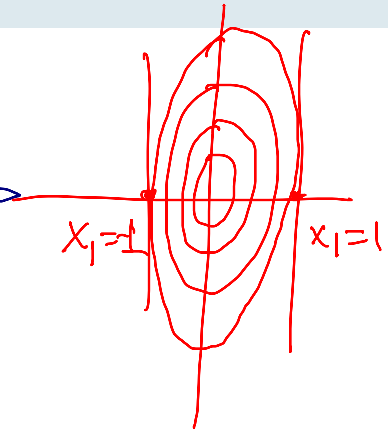
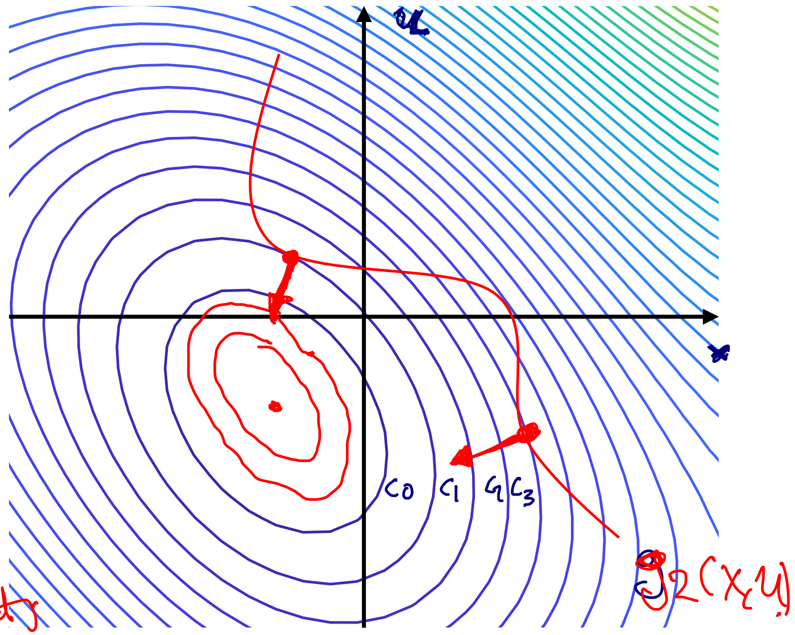
Recap static optimization

Optimization under constraints:



$$\begin{array}{ll} \min_{x,u} & g_1(x,u) \\ \text{s.t.} & g_2(x,u) = 0 \end{array}$$

$g_1(x,u) = c$



Necessary conditions for optimality:

- ∇g_1 points in the same direction as ∇g_2
- $g_2(x,u) = 0$

λ Lagrange multipliers

#λ = # constraints

Lagrangian: $\mathcal{L}(x,u,\lambda) = \underbrace{g_1(x,u)}_{\text{cost}} + \lambda^T \underbrace{g_2(x,u)}_{\text{constraints}}$

Lagrange multipliers

- $\frac{\partial \mathcal{L}}{\partial x} = 0, \ \frac{\partial \mathcal{L}}{\partial u} = 0, \ \frac{\partial \mathcal{L}}{\partial \lambda} = 0$
- $\frac{\partial^2 \mathcal{L}}{\partial x^2} > 0, \ \frac{\partial^2 \mathcal{L}}{\partial u^2} > 0$

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial u} \\ \frac{\partial \mathcal{L}}{\partial \lambda} \end{bmatrix} = 0 \qquad \begin{bmatrix} \frac{\partial g_1(x,u)}{\partial x} + \lambda \frac{\partial g_2(x,u)}{\partial x} \\ \frac{\partial g_1(x,u)}{\partial u} + \lambda \frac{\partial g_2(x,u)}{\partial u} \\ g_2(x,u) = 0 \end{bmatrix} = 0$$

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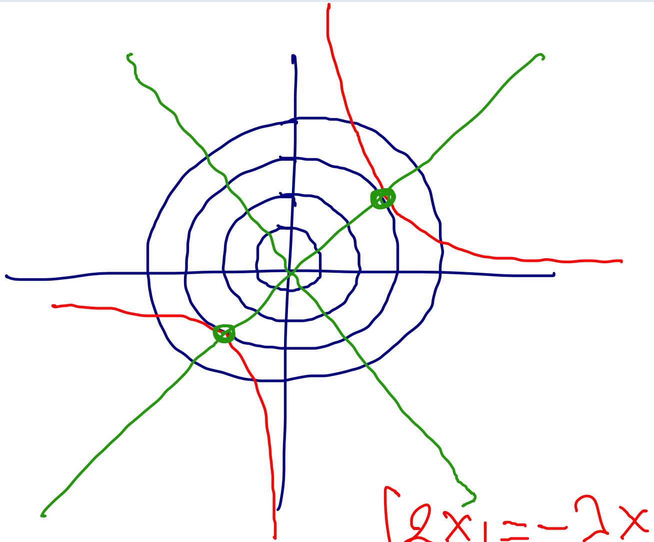
Static optimization

$$\begin{array}{ll} \min_{x_1,x_2} & (x_1^2 + x_2^2) \\ \text{s.t.} & \boxed{x_1 x_2 - 1 = 0} \end{array} \longrightarrow \text{1 constraint}$$



$$\mathcal{L} = x_1^2 + x_2^2 + \lambda (x_1 x_2 - 1)$$

$$\begin{bmatrix} \frac{\partial g_1(x_1,x_2)}{\partial x_1} + \lambda \frac{\partial g_2(x_1,x_2)}{\partial x_1} \\ \frac{\partial g_1(x_1,x_2)}{\partial x_2} + \lambda \frac{\partial g_2(x_1,x_2)}{\partial x_2} \\ g_2(x_1,x_2) = 0 \end{bmatrix} \begin{array}{l} \frac{\partial \mathcal{L}}{\partial x_1} \\ \frac{\partial \mathcal{L}}{\partial x_2} \end{array} = 0 \quad \Rightarrow \quad \begin{cases} 2x_1 + \lambda x_2 = 0 \\ 2x_2 + \lambda x_1 = 0 \\ x_1 x_2 - 1 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} (x_1 - x_2)(x_1 + x_2) = 0 & (1) \\ x_1 x_2 - 1 = 0 & (2) \\ 2x_1 + \lambda x_2 = 0 \end{cases}$$

$$\begin{cases} 2x_1 = -\lambda x_2 \\ 2x_2 = -\lambda x_1 \end{cases} \Rightarrow \frac{x_1}{x_2} = +\frac{x_2}{x_1} \Rightarrow x_1^2 - x_2^2 = 0$$

(1) 2 solutions

$x_1 = x_2 \Rightarrow x_2^2 = 1 \Rightarrow x_2 = \pm 1 = x_1$

$x_1 = -x_2 \Rightarrow x_2^2 + 1 = 0$ no real solutions

Maximum principle – no final time constraint

cost

Optimization Problem (OP1)

Minimize $\displaystyle\int_0^{t_f} \overbrace{L(x(t), u(t))}^{\text{Trajectory cost}}\, dt + \overbrace{\phi(x(t_f))}^{\text{Final cost}}$

where

$x(t) \in R^n, \quad u(t) \in U \subseteq R^m \rightarrow$ inequality constraint

$\boxed{\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0}$

$0 \leq t \leq t_f,$ given fixed end-time t_f

Lagrange multipliers

Theorem 18.2 of Glad/Ljung Assume that the (OP1) has a solution $\{u^*(t), x^*(t)\}$.
Then

$$H(x, u, \lambda) = L(x, u) + \lambda^T(t) f(x, u)$$

$$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \leq t \leq t_f,$$

where $\lambda(t)$ solves the **adjoint equation**

$$H_x = \frac{\partial H}{\partial x} = \begin{pmatrix} \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} & \dots \end{pmatrix}$$

$$\frac{d\lambda}{dt} = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \text{with} \quad \lambda(t_f) = \phi_x^T(x^*(t_f))$$

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Sketchy proof (Hamiltonian)

Minimize $\displaystyle\int_{t_0}^{t_f} \overbrace{L(x(t),u(t))}^{\text{Trajectory cost}}\, dt + \overbrace{\phi(x(t_f))}^{\text{Final cost}}$

where

$x(t) \in R^n, \quad u(t) \in U \subseteq R^m$

$\dot{x}(t) = f(x(t),u(t)), \qquad x(t_0) = x_0$

$t_0 \leq t \leq t_f,$ given fixed end-time t_f

$f - \dot{x} = 0$

Optimal Control Problem

$$\min_u J = \min_u \left\{ \phi(x(t_f)) + \int_{t_0}^{t_f} L(x,u)\, dt \right\}$$

$$\text{subject to} \quad \dot{x} = f(x,u), \quad x(t_0) = x_0$$

Functions of time the constraint is satisfied over the assumed period of time

$$J = \phi(x(t_f)) + \int_{t_0}^{t_f} \left(\boxed{L(x,u) + \lambda^T (f} - \dot{x})\right) dt$$

$$H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$$

$$= \phi(x(t_f)) - \left[\lambda^T x\right]_{t_0}^{t_f} + \int_{t_0}^{t_f} \left(H + \dot{\lambda}^T x\right) dt$$

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Sketchy proof (Calculus of variation)

Variation of J:

$$\delta J = \left[\left(\frac{\partial \phi}{\partial x} - \lambda^T \right) \delta x \right]_{t=t_f} + \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \delta x + \frac{\partial H}{\partial u} \delta u \right] dt$$

Necessary conditions for local minimum ($\delta J = 0$)

$$\boxed{\lambda(t_f)^T} = \boxed{\frac{\partial \phi}{\partial x}}\bigg|_{t=t_f} \qquad \boxed{\dot{\lambda}^T = -\frac{\partial H}{\partial x}} \qquad \underbrace{\frac{\partial H}{\partial u} = 0}$$

H_x

- λ specified at $t = t_f$ and x at $t = t_0$
- Two Point Boundary Value Problem (TPBV)
- For sufficiency $\frac{\partial^2 H}{\partial u^2} \geq 0$

Summary of the approach

Performance, cost function $$J(x_0) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x,u)\,dt$$

System dynamics $$\dot{x} = f(x,u),\ x(t_0) = x_0$$

No final-time constraint but final time is a free variable

Hamiltonian $$H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$$

State equation $$\dot{x} = f(x,u), \quad x(t_0) = x_0 \qquad \textbf{(1)}$$

Co-state, adjoint equation $$\dot{\lambda} = H_x^T(x,u,\lambda), \quad \lambda(t_f) = \phi_x^T[x(t_f)] \qquad \textbf{(2)}$$

$$\phi_x = \left[\frac{\partial \phi}{\partial x_1} \cdots \frac{\partial \phi}{\partial x_n}\right]$$

Hamiltonian minimization with respect to u $$\min_{u \in U} H(u) \qquad \textbf{(3)}$$

We can often first eliminate the control input u(t) by (3)

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Remarks

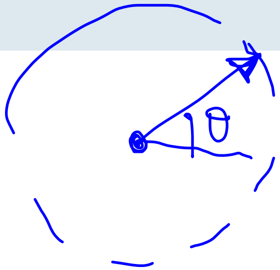
- The Maximum Principle gives **necessary** conditions
- A pair $(u^*(\cdot), x^*(\cdot))$ is called **extremal** the conditions of the Maximum Principle are satisfied.
- Many extremals can exist.
- The maximum principle gives all possible candidates.
- However, there might not exist a minimum!

Example

Minimize $x(1)$ when $\dot{x}(t) = u(t)$, $x(0) = 0$ and $u(t)$ is free

Why doesn't there exist a minimum?

Example 1



$$\min_{u:[0,t_f)\to U} \ -x_1(T)$$

$$\dot{x}_1 = v(x_2) + u_1$$

$$\dot{x}_2 = u_2$$

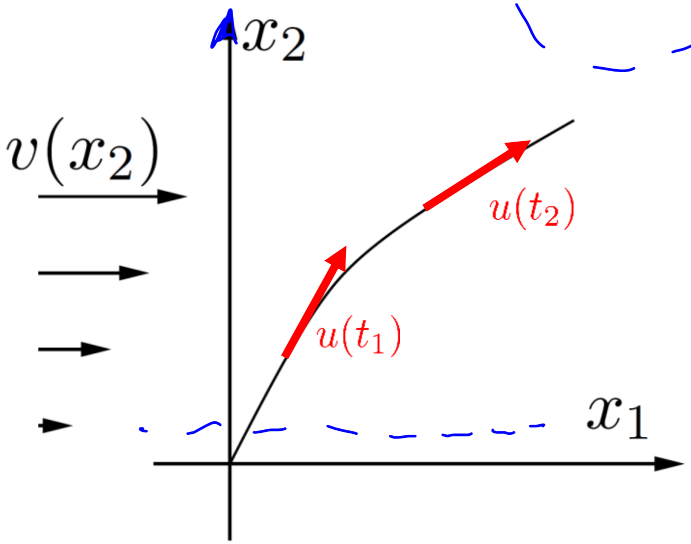
$$x_1(0) = 0$$

$$x_2(0) = 0$$

$$\boxed{u_1^2 + u_2^2 = 1}$$

$$\min \int_0^{t_f} \mathcal{L}(x,u) + \phi(x(t_f))$$

$$\phi(x(t_f)) = -x_1(T)$$



- Speed of water $v(x_2)$ $= x_2$ in x_1 direction with $\boxed{\frac{\partial v(x_2)}{\partial x_2} = 1}$
- Move (sail) maximum distance in x_1-direction in fixed time T
- Rudder angle control: $u \in U := \{(u_1, u_2) : \quad u_1^2 + u_2^2 = 1\}$

Example 1

Hamiltonian:

$$H = 0 + \lambda^T(t) f = [\lambda_1(t)\ \lambda_2(t)] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \lambda_1(v(x_2) + u_1) + \lambda_2 u_2$$

H is not a function of x_1.

Adjoint equation:

$$\begin{bmatrix} \dot\lambda_1 \\ \dot\lambda_2 \end{bmatrix} = \begin{bmatrix} -\partial H/\partial x_1 \\ -\partial H/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -v'(x_2)\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

1

$\dot\lambda_1 = 0 \Rightarrow \lambda_1(t) = -1, \forall t$

$\dot\lambda_2 = -\lambda_1 \Rightarrow \dot\lambda_2 = 1$

with boundary conditions

$$\begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \end{bmatrix} = \begin{bmatrix} \partial\phi/\partial x_1|_{x=x^*(t_f)} \\ \partial\phi/\partial x_2|_{x=x^*(t_f)} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$\phi(x_1) = -x_1$

$\lambda_1(T) = -1$

$\lambda_2(T) = 0$

$\dot\lambda_2 = 1 \Rightarrow \lambda_2 = t + C$

$\lambda_2 = t$

$\lambda_2 = t - T$

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Example 1

$\min\limits_{\|b\|=1} a^T b$ ~ when $\theta = \pi$ $\qquad a^T b\big|_{\theta=\pi} = -\|a\|$

$a^T b = \|a\| \cos\theta$

Solution of the co-state $\lambda_1(t) = -1, \quad \lambda_2(t) = t - T.$

Optimality: Control signal should solve

$$\min_{u_1^2+u_2^2=1} \lambda_1(v(x_2)+u_1)+\lambda_2 u_2$$

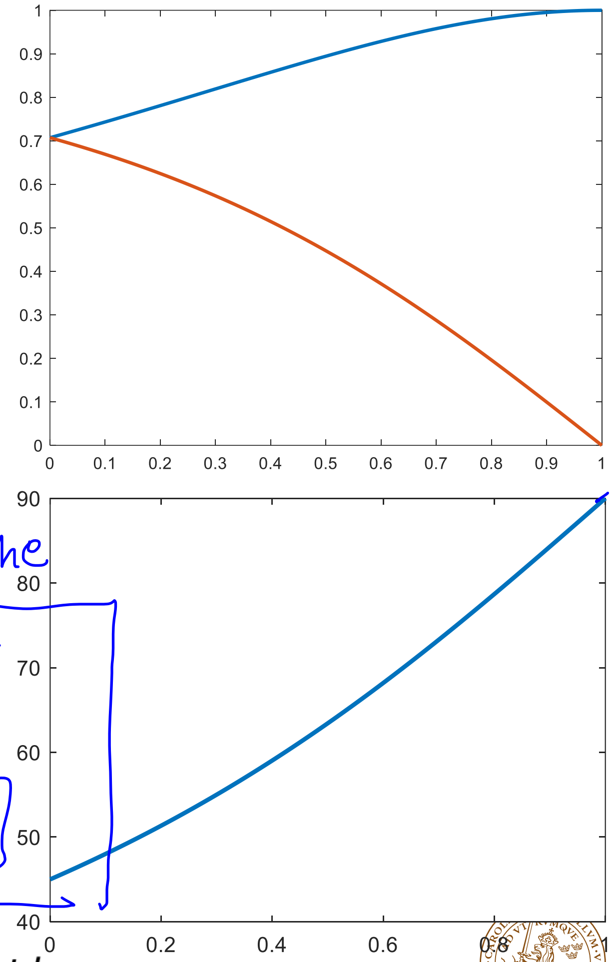
Minimize $\lambda_1 u_1 + \lambda_2 u_2$ so that (u_1, u_2) has length 1 $\qquad \min\limits_{u_1^2+u_2^2=1} [\lambda_1, \lambda_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ unit

$$u_1(t) = -\frac{\lambda_1(t)}{\sqrt{\lambda_1^2(t)+\lambda_2^2(t)}}, \quad u_2(t) = -\frac{\lambda_2(t)}{\sqrt{\lambda_1^2(t)+\lambda_2^2(t)}}$$

when u is in the opposite direction with $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$$u_1(t) = \frac{1}{\sqrt{1+(t-T)^2}}, \quad u_2(t) = \frac{T-t}{\sqrt{1+(t-T)^2}}$$



See fig 18.1 for plots

Remark: It can be shown that this optimal control problem has a minimum. Hence it must be the one we found, since this was the only solution to MP

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