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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 7: Indirect Lyapunov's method and Input-Output Stability

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- Lab 1 today
- Lecture 28 Nov Monday
  → 2 Dec Friday

Outline

- Lyapunov Analysis for Linearized systems   Indirect Lyapunov method.
- Indirect Lyapunov's Method
- Small-gain theorem
- Circle Criterion (the point -1/k is replaced by a cycle)

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Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

or

1. Choose an arbitrary symmetric, positive definite matrix Q.
2. Find P that satisfies Lyapunov equation

$Ax = b$

$$\boxed{PA + A^T P = -Q}$$

and verify that it is positive definite.

Lyapunov function: $\boxed{V(x) = x^T P x}$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = -x^T Q x$$

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Lyapunov analysis for Linear systems

$\lambda_{min}(M)\|x\|^2 \leq \underbrace{x^T M x} \leq \lambda_{max}(M)\|x\|^2$

$$\dot{x} = Ax = \underbrace{\begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. Let $Q = I_2$
2. Solve P from the Lyapunov equation

$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for p_{11}, p_{12} and p_{22} gives

$\dot{V} = -x^T Q x$

$\dot{V} \leq \underbrace{-\lambda_{min}(Q)\|x\|^2}$

$$\begin{aligned} 2p_{11} &= -1 \\ -4p_{12} + 4p_{11} &= 0 \\ 8p_{12} - 6p_{22} &= -1 \end{aligned} \implies \overbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}^{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

$\boxed{\lambda_{min}(P)\|x\|^2 \leq V \leq \lambda_{max}(P)\|x\|^2}$

Exponential stability theorem.

$\det(P) > 0$

p_{11} or $p_{22} > 0$

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Lyapunov's indirect method

Theorem Consider

$$\dot{x} = f(x)$$

Assume that $f(0) = 0$. Linearization

$$\dot{x} = Ax + g(x)\,, \qquad \boxed{||g(x)|| = o(||x||) \text{ as } x \to 0\,.}$$

(1) Re $\lambda_k(A) < 0,\ \forall k \quad \Rightarrow \quad x = 0$ locally asympt. stable

(2) $\underline{\exists k : \text{Re}\lambda_k(A) > 0} \quad \Rightarrow \quad x = 0$ unstable

Lyapunov function candidate: $V(x) = x^T P x$

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} \qquad A^T \underline{P} + PA = -Q$$

Jacobian

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\dot{x} = \underbrace{\frac{\partial f}{\partial x}\Big|_{x=0}}_{A} x + \underbrace{\left[f(x) - \frac{\partial f}{\partial x}\Big|_{x=0} x \right]}_{g(x)}$$

Lyapunov's indirect method

– Choose Q and solve $PA + A^TP = -Q$.

Lyapunov function candidate: $V(x) = x^TPx$

Differentiating $\dot{V}(x)$ *along system's trajectories* $\dot{x} = Ax + g(x)$ $= f(x)$

nonlinear part

$\dot V = x^TP\dot x + \dot x^TPx$

$$\begin{aligned}\dot{V}(x) &= x^TPf(x) + f^T(x)Px \\ &= x^TP[Ax + g(x)] + [x^TA^T + g^T(x)]Px \\ &= x^T\underbrace{(PA + A^TP)}_{-Q}x + 2x^TPg(x) = -x^TQx + \boxed{2x^TPg(x)}\end{aligned}$$

$A \qquad A^T$

$\|a^Tb\| \le \|a\|\|b\|$

$$\boxed{x^TQx \ge \lambda_{\min}(Q)\|x\|^2}$$

$\dot V \le -\lambda_{\min}(Q)\|x\|^2 + 2\|x\| \, \|P\| \, \|g(x)\|$

$\gamma\|x\|$

and for all $\gamma > 0$ there exists $r > 0$ such that

$\dot V$ negative

$$\boxed{\|g(x)\| < \gamma\|x\|, \qquad \forall\|x\| < r}$$

$\dot V \le -\theta\|x\|^2$

Thus, choosing γ sufficiently small gives

$$\boxed{\dot{V}(x) \le -\left(\lambda_{\min}(Q) - 2\gamma\lambda_{\max}(P)\right)\|x\|^2 < 0}$$

$\lambda_{\min}(Q) - 2\gamma\lambda_{\max}P > 0$

$\gamma < \dfrac{\lambda_{\min} Q}{2\lambda_{\max} P}$

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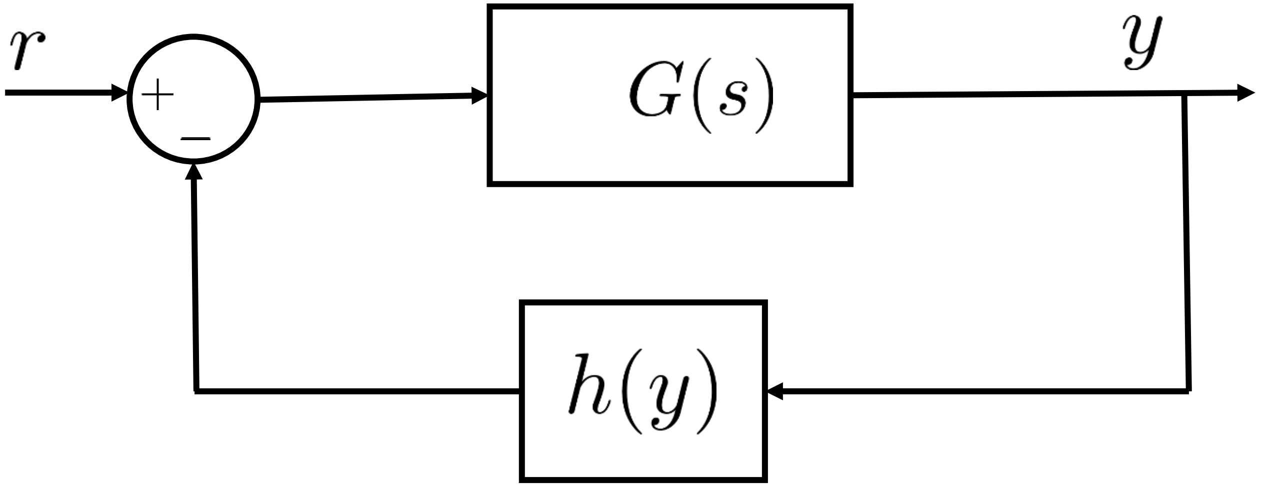
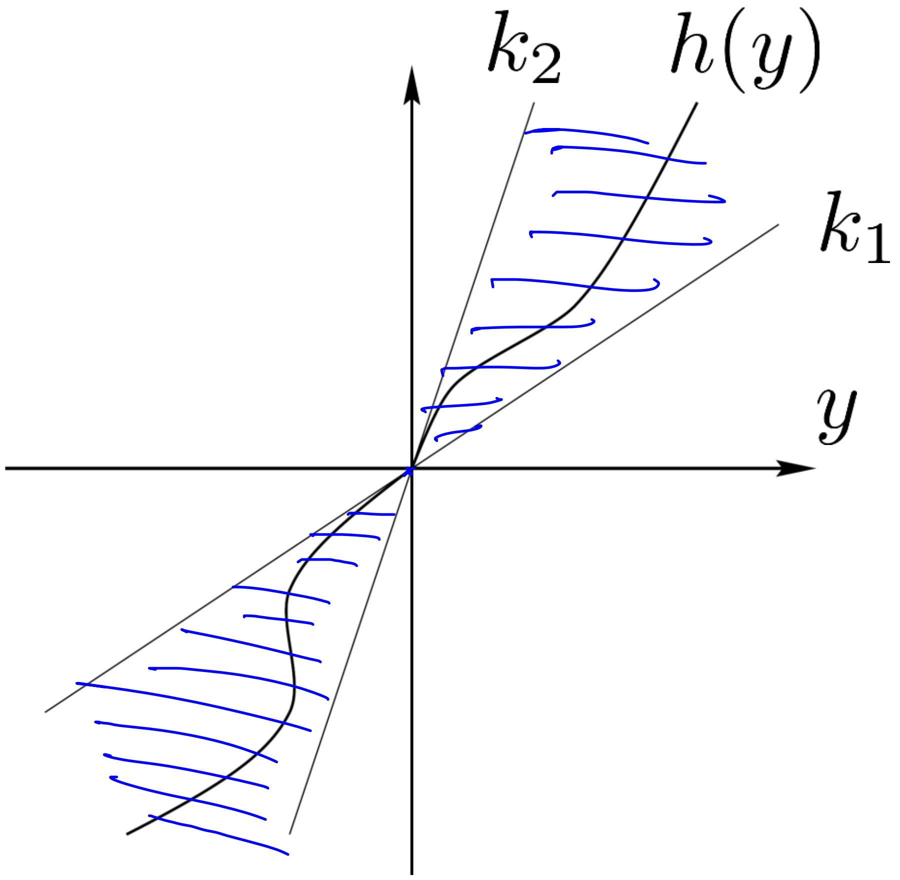
Feedback form where the nonlinearity is in a block





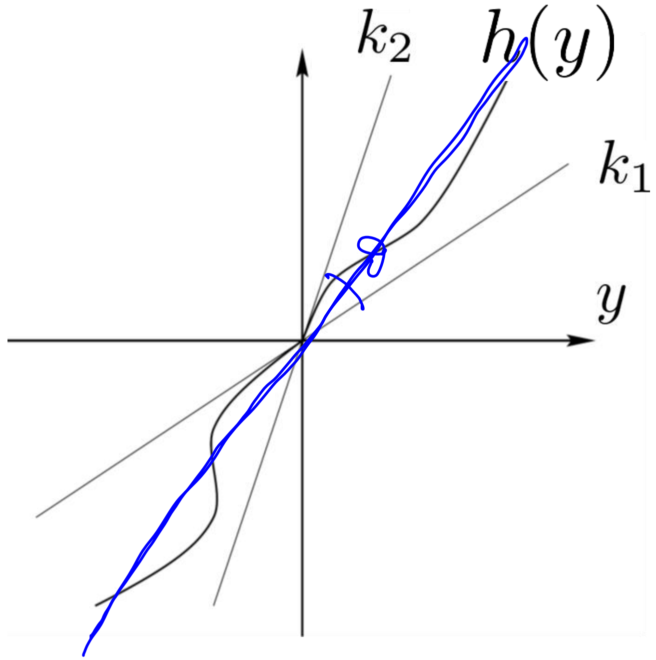
$$\dot{x} = Ax + g(x) = f(x)$$

$$= Ax + Bg(x)$$

$\Downarrow$ I

Sector nonlinearity

$\dot{y} = h(y)$



Gain.

$y \rightarrow$ [ ] $\rightarrow h(y)$

Bound in sector function $h \in \text{sector}[k_1, k_2]$

- $h(y)$ continuous wrt y
- $h(0) = 0$
- $k_1 \leq \frac{h(y)}{y} \leq k_2, \forall y \neq 0$ or equivalently $k_1 y^2 \leq y h(y) \leq k_2 y^2 \; \forall \; y$

inequality $\rightarrow$]

Other cases:

- $h \in \text{sector}(k_1, k_2]$ $\Rightarrow$ $k_1 < \frac{h(y)}{y} \leq k_2$
- $h \in \text{sector}[k_1, \infty)$ $\Rightarrow$ $k_1 \leq \frac{h(y)}{y}$
- $h \in \text{sector}[0, \infty)$ (first and third quadrant)

$h \in [0, \infty)$

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Signal norms and spaces

- A signal $x(t)$ is a function from $\mathbf{R}^+$ to $\mathbf{R}^d$. $x(t) \in \mathbb{R}^d$ $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$
- A signal norm is a way to measure the size of $x(t)$ in long run:

  2-norm (energy norm): $\|x\|_2 = \sqrt{\int_0^\infty \|x(t)\|^2 dt}$

  sup-norm: $\|x\|_\infty = \sup_{t \in \mathbf{R}^+} |x(t)|$

- The space of signals with $\|x\|_2 < \infty$ is denoted $\mathcal{L}_2$. $x(t) \in \mathcal{L}_2$
- The space of signals with $\|x\|_\infty < \infty$ is denoted $\mathcal{L}_\infty$.
- $x(t) \in \mathcal{L}_2$ corresponds to bounded energy signals.
- $x(t) \in \mathcal{L}_\infty$ corresponds to bounded signals.

Equivalent expression in frequency domain

$$\|x\|_2^2 = \int_0^\infty \|x(t)\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|X(j\omega)\|^2 d\omega$$

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Gain of a system $\mathcal{L}_2$

$u \longrightarrow \boxed{S} \longrightarrow y$

Gain of S: $\gamma(S) = \sup_{u\in\mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u\in\mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$

S is bounded-input bounded-output (BIBO) stable if $\gamma(S) < \infty$.

$u \longrightarrow \boxed{\alpha} \longrightarrow y$

Static linear relation

$$\gamma(\alpha) = \sup_{u\in\mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u\in\mathcal{L}_2} \frac{|\alpha|\|u\|_2}{\|u\|_2} = |\alpha|$$

$u \longrightarrow \boxed{G(s)} \longrightarrow y$

Stable Linear System

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = C(sI - A)^{-1}B + D$$

$y =$

$$\gamma(G) = \sup_{u\in\mathcal{L}_2} \frac{\|Gu\|_2}{\|u\|_2} = \boxed{\sup_{\omega\in(0,\infty)} |G(j\omega)|}$$

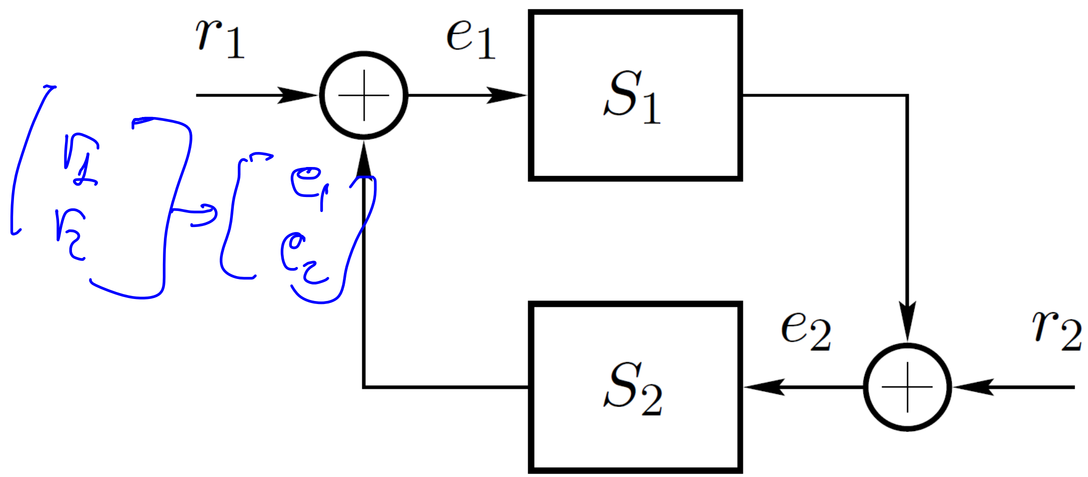
$u \longrightarrow \boxed{h(\cdot)} \longrightarrow y$

Static Nonlinearity

$$h(\cdot) \in [-K \quad K]$$

$$\boxed{\gamma(h) = \sup_{u\in\mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = K}$$

The Small-Gain Theorem



$\gamma(S_1) < \infty \quad , \quad \gamma(S_2) < \infty$

Theorem

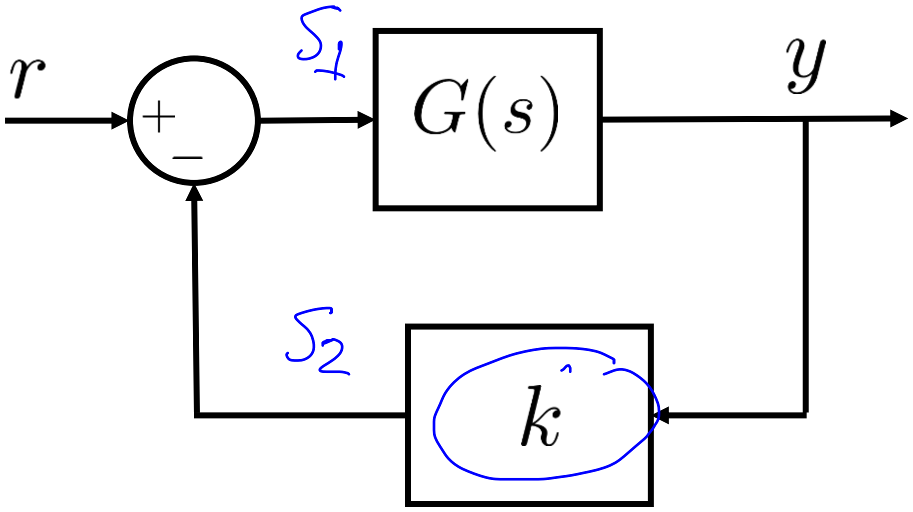
Assume S_1 and S_2 are BIBO stable. If

$$\gamma(S_1)\gamma(S_2) < 1$$

then the closed-loop map from (r_1, r_2) to (e_1, e_2) is BIBO stable.

$\gamma(G(s)) = \sup_{\omega \in (0,\infty)} |G(j\omega)|$

$\sup_{\omega \in (0,\infty)} |G(j\omega)| \, K < 1.$



$$\sup_{\omega \in (0,\infty)} |G(j\omega)| < \frac{1}{k}$$

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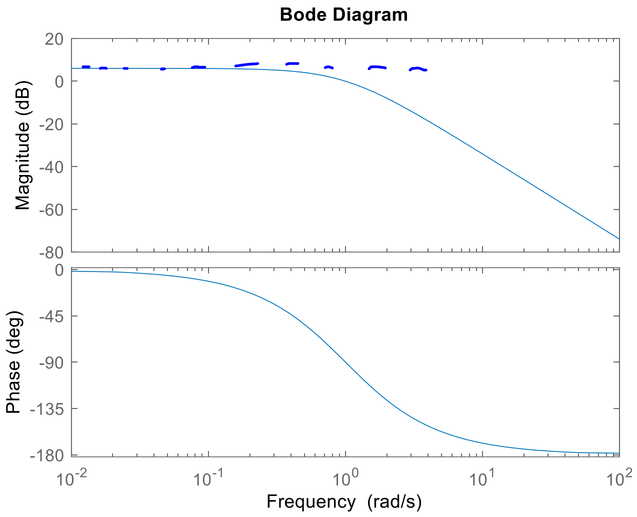
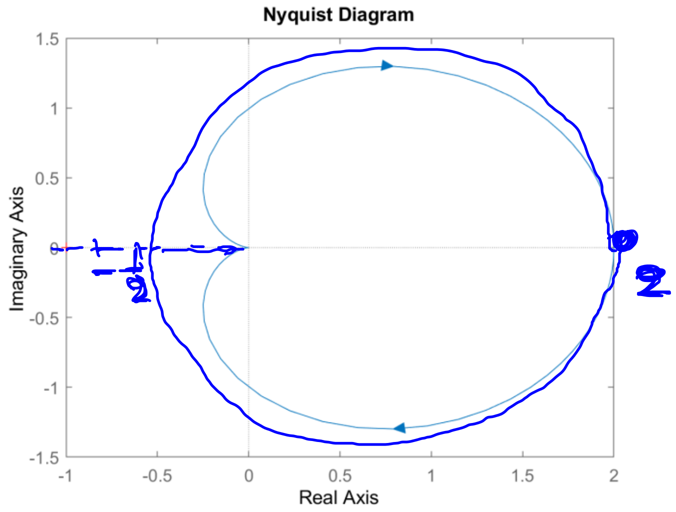
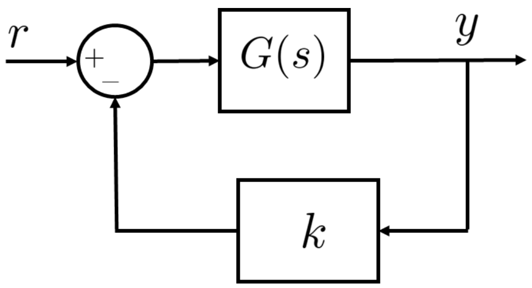
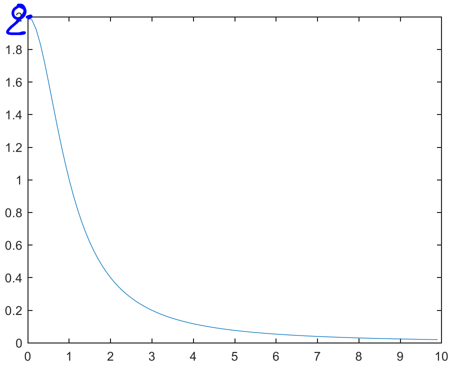
Small-Gain Theorem is conservative







$$G(s) = \frac{2}{(s+1)^2}$$



Small gain Theorem

$\gamma_a \cdot \gamma_k < 1$

$$\sup_{\omega\in(0,\infty)} |G(j\omega)||k| < 1$$

$$\boxed{|k| < \frac{1}{2}}$$

$$\rightarrow G(s) = \frac{2}{s^2+2s+1}$$

$$\rightarrow G(j\omega) = \frac{2}{1-\omega^2+2j\omega} = \frac{2(1-\omega^2)}{(1-\omega^2)^2+4\omega^2} - j\frac{4\omega}{(1-\omega^2)^2+4\omega^2}$$

$$\boxed{|G(j\omega)| = \frac{2}{\sqrt{(1-\omega^2)^2+4\omega^2}}}$$

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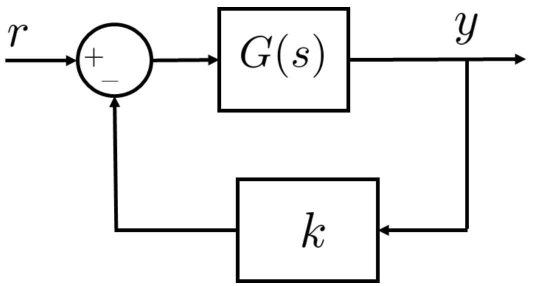
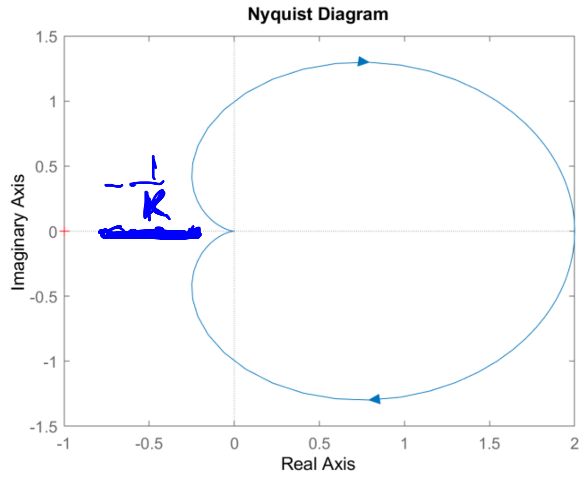
Nyquist criterion



- $P_{CL_{\mathrm{Re}>0}} = \mathrm{N}(-1/k, 0) + P_{OL_{\mathrm{Re}>0}}$



- Given a stable open loop system, the closed loop is stable if the Nyquist plot of the open loop system does not encircle the point (-1/k,0) in the clockwise direction.

$k_1 \le \frac{h(y)}{y} \le k_2 \rightarrow k_1 = k_2$

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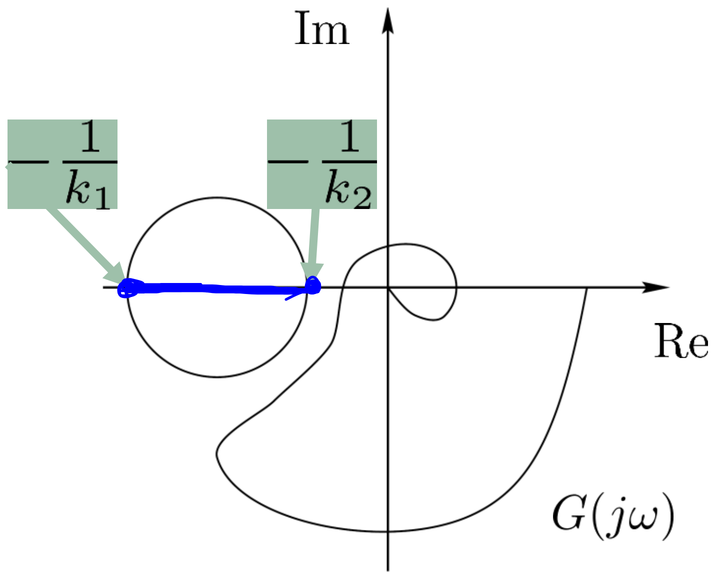
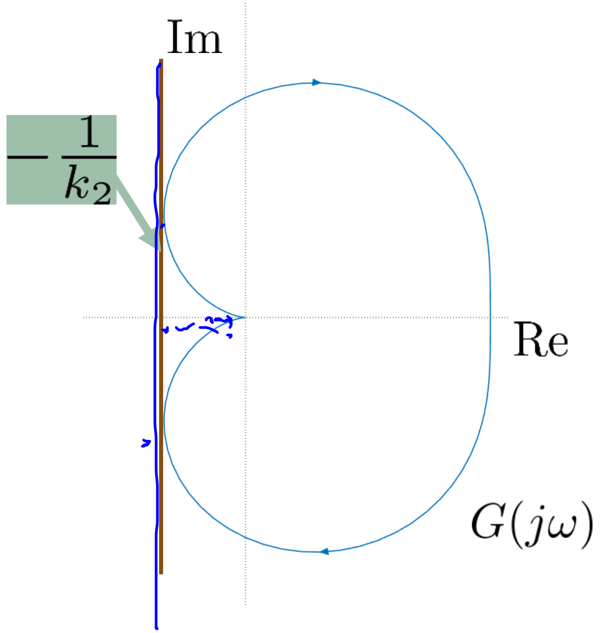
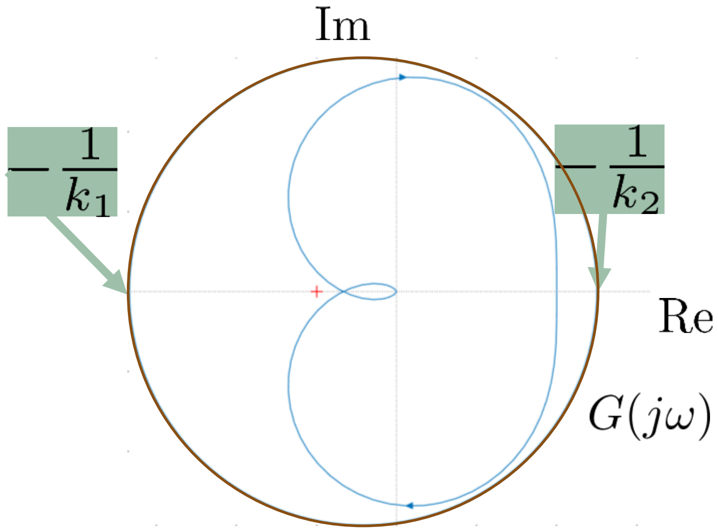
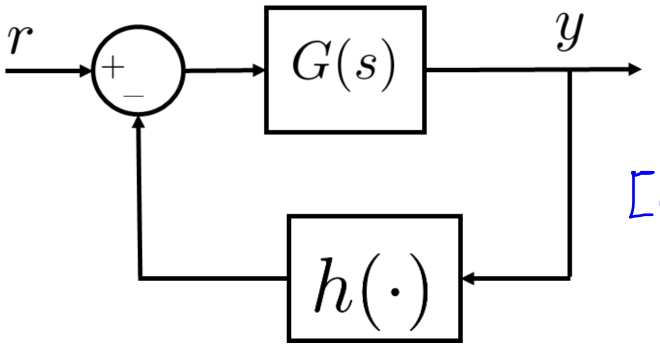
Circle criterion





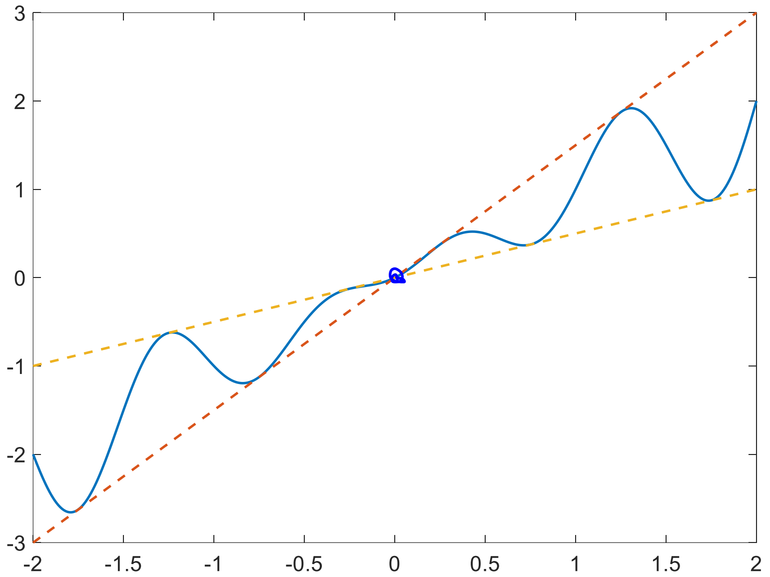


- $G(s)$ **stable**
  - $0 < k_1 < k_2$: The Nyquist curve of $G(s)$ does not intersect or encircle the circle defined by the points $-1/k_1$ and $-1/k_2$
  - $[0, k_2]$ $0 = k_1 < k_2$: The Nyquist curve of $G(s)$ stays to the right of the line Re $s = -1/k_2$
  - $k_1 < 0 < k_2$: The Nyquist curve of $G(s)$ stays inside the circle



$h(x) \in [k_1, k_2]$

Circle criterion – Example 1



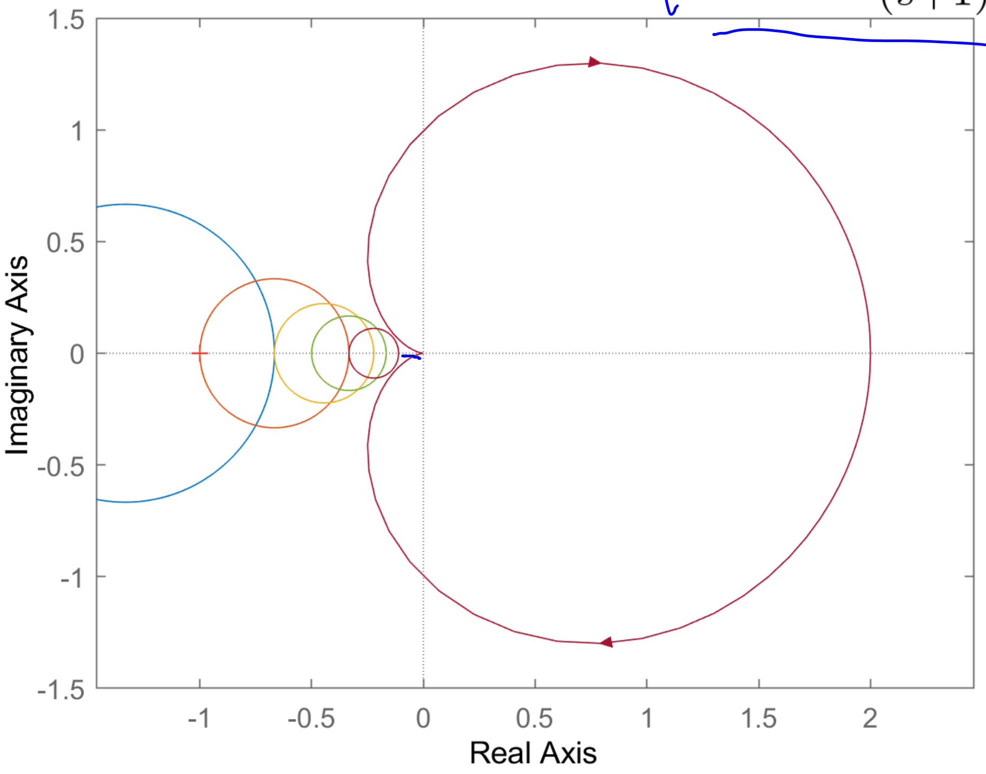
$$\begin{cases} \dot{x} = -x + r - h(y) \\ \dot{y} = -y + 2x \end{cases} \Rightarrow$$

$sX = -X + r - h(y)$

$sY = -Y + 2X \Rightarrow (s+1)Y = 2X$

$h(y) = y + \frac{1}{2} y \sin(y)$ $\quad (s+1)X = r - h(y) \quad (*)$

$$\boxed{G(s) = \frac{2}{(s+1)^2}}$$



$h(y) \in [0.5, 1.5]$

1) Find the sector

$h(y) = y + \frac{1}{2}\sin(y)\, y \Rightarrow h(0) = 0$

$\Rightarrow \frac{h(y)}{y} = 1 + \frac{1}{2}\sin(y)$

2) Apply Circle Criterion depending on the sector

$= 1 + \frac{1}{2}\sin(y)$

$1 + \frac{1}{2}(-1) \leq \sin y \leq \frac{1}{2}(1) + 1$

$0.5 \leq h(y) \leq 1.5$

Circle criterion – Example 2

$$\begin{cases} \dot{x} = -x + r - f(y,t) \\ \dot{y} = -y + 2x \end{cases}$$

$$\boxed{f(y,t) = (3 - 4e^{-2t})y}$$

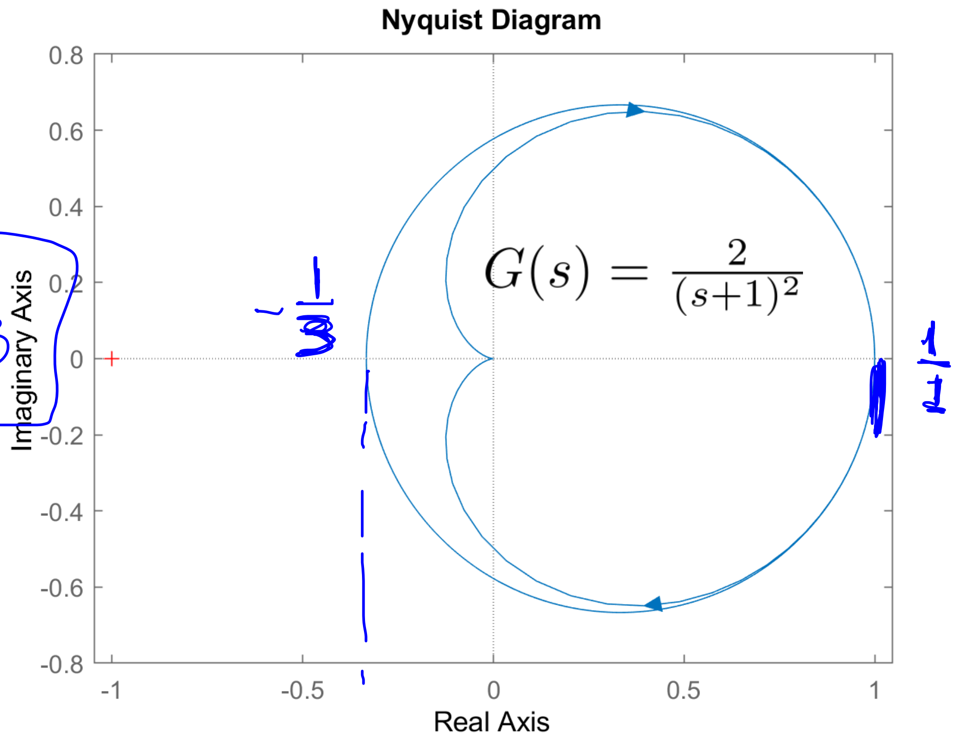
1) Find the sector

$f(0,t) = 0$

$\frac{f(y,t)}{y} = 3 - 4e^{-2t}$

$\boxed{-1 \leq \frac{f(y,t)}{y} < 3}$

2) Apply Circle Criterion depending on the sector



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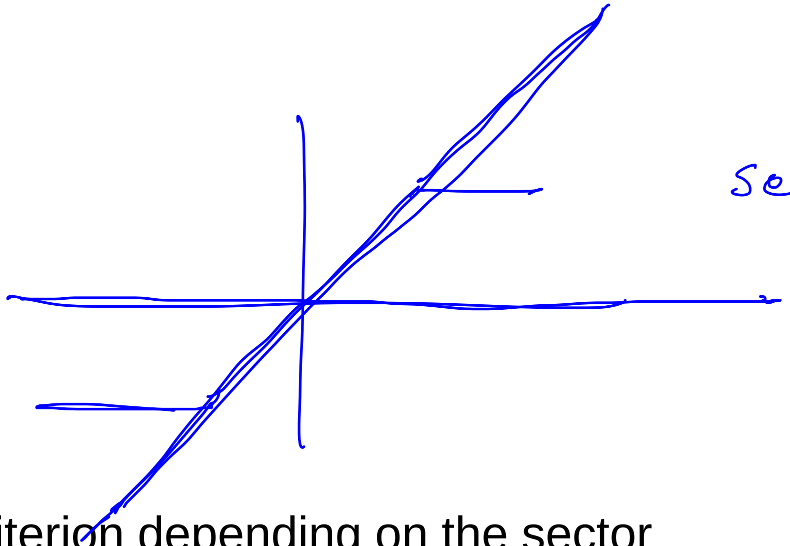
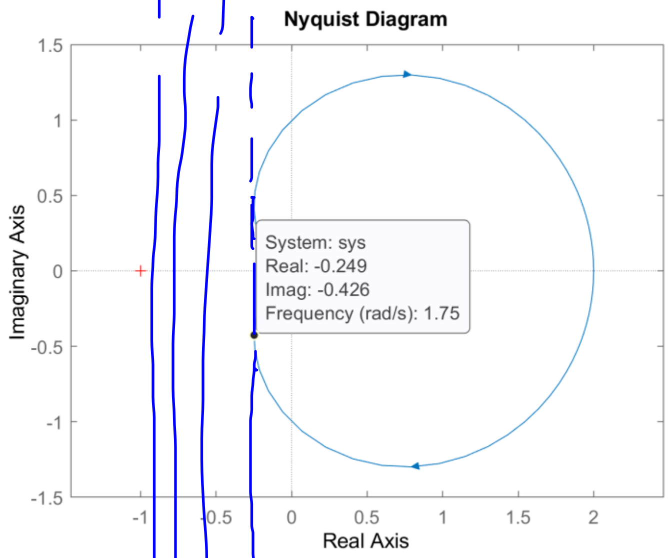
Circle criterion – Example 3

$$\begin{cases} \dot{x} = -x + r - \text{sat}(y) \\ \dot{y} = -y + 2x \end{cases}$$

1) Find the sector



Sector(0, [illegible]]



2) Apply Circle Criterion depending on the sector

$$G(j\omega) = \frac{2}{1-\omega^2+2j\omega} = \frac{2(1-\omega^2)}{(1-\omega^2)^2+4\omega^2} - j\frac{4\omega}{(1-\omega^2)^2+4\omega^2}$$